VI. Module 6: EITM and Simultaneity

A Theoretical Basis for Simultaneity: The Role of Expectations, Learning, and Social Interaction

- Example: The Boomerang Effect
  - Question: What are the consequences for the rational expectations equilibrium if there exists simultaneous **interaction** among agents in the context of learning dynamics?
  - The inaccurate forecasts by the public (less attentive) may affect the forecasts of the highly attentive (policymakers) – the **Boomerang Effect.**

- In previous literature (rational expectations), the expectations are conditioned on all of the information available to the decision makers.

- Limitations: Strong Assumption – Not Realistic.

- Relax this strong assumption and assume agents are boundedly rational.

- In the adaptive learning literature an assumption is typically made that agents have *Homogeneous Expectations*, heterogeneous expectations, or rational heterogeneous expectations.

What is Missing in the adaptive learning literature?

- All agents are assumed to learn ***independently***. There is ***no interaction*** among agents.

- Here we assume interaction among (between) agents.

- Two groups of agents: Less Attentive (Group L) and High Attentive (Group H).

- Two groups are evenly distributed.

- Group L ***learns*** from the expectations of Group H with "measurement error". ***(Interaction Assumption)***

2 Interaction Scenarios:

Model 1: *The Case of* ***Homogeneous Information*** *Sets*

- Both groups of agents are forming their expectations with the ***same (full)*** information set.

- Results:

  - Group L ***relies*** on the expectations of Group H for a ***limited time***.

Model 2: *The Case of* ***Heterogeneous Information*** *Sets*

- Group L has only a ***subset*** of relevant information.

- Results:

    - Both groups learn the ***restrictive perceptions equilibrium (RPE)***.

    - Group L finds it necessary to ***rely on the expectations*** of Group H.

- ***Boomerang Effect:*** the inaccurate forecasts of Group L confound Group H's convergence to the rational expectations equilibrium and predictive accuracy.

Details of the Model

The True Model (Cobweb Variety):

$$y_t = \alpha + \beta E^*_{t-1} y_t + \gamma' w_{t-1} + \eta_t \qquad (1)$$

Notation:

y_t = the endogenous variable;

$E^*_{t-1} y_t =$ the average expectations of y_t at time t-1;

w_{t-1} = a vector of exogenous variables.

The Rational Expectations Equilibrium (REE):

$$y_t = \frac{\alpha}{1-\beta} + \frac{\gamma'}{1-\beta} w_{t-1} + \eta_t$$

Questions:

- Is this REE learnable?
- Does the equilibrium change after introducing the interaction assumption?

Examples of Cobweb Applications:

1. Cobweb Model (Muth 1961):

- *Supply-Demand* model in an isolated market.
- Firms produce based on the *expected future price*

$$d_t = m_I + m_p p_t + v_{1t}$$

$$s_t = r_I + r_p p_t^e + r'_w w_{t-1} + v_{2t}$$

Assuming market clearing, the reduced form of the price function is:

$$p_t = \mu + \alpha p_t^e + \delta' w_{t-1} + \eta_t$$

---

2. Lucas Aggregate Supply Model (Lucas 1973)

$$q_t = \overline{q} + \pi\left(p_t - p_t^e\right) + \varsigma_t$$
$$m_t + v_t = p_t + q_t$$
$$v_t = \mu + \gamma' w_{t-1} + \xi_t$$
$$m_t = \overline{m} + u_t + \rho' w_{t-1}$$

The reduced form of the price function is:

$$p_t = \Sigma_0 + \Sigma_1 p_t^e + \Sigma_2 w_{t-1} + \varepsilon_t$$

Learning Dynamics:

Perceived Law of Motion (PLM)

Group H:

$$y_t = a_{H,t-1} + b'_{H,t-1} w_{t-1} + v_t \qquad (2)$$

Group H's forecast of y_t at time t-1:

$$E^*_{t-1} y_{H,t} = a_{H,t-1} + b'_{H,t-1} w_{t-1} \qquad (3)$$

Group L:

$$y_t = a_{L,t-1} + b'_{L,t-1} w_{t-1} + c_{L,t-1}\hat{y}_{t-1} + v_t \quad (4)$$

and

$$\hat{y}_{t-1} = E^{*}_{t-1} y_{H,t} + \tilde{e}_{L,t-1}, \quad (5)$$

where $\tilde{e}_{L,t-1}$ is the ***measurement error*** of observing the expectations of Group H.

Group L's forecast:

$$E^{*}_{t-1}\, y_t = a_{L,t-1} + b'_{L,t-1} w_{t-1} + c_{L,t-1}\hat{y}_{t-1} \quad (6)$$

Actual Law of Motion (ALM)

- ***Actual Law of Motion*** (ALM) describes the stochastic process of y_t followed by the economy if forecasts are made under the fixed rule given by the PLM.

- Since both groups are ***evenly distributed***, the average expectations is:

$$E^*_{t-1}y_t = \frac{E^*_{t-1}y_{H,t} + E^*_{t-1}y_{L,t}}{2} \qquad (8)$$

  - Remember, the true model (1) is:

$$y_t = \alpha + \beta E^*_{t-1}y_t + \gamma' w_{t-1} + \eta_t$$

  - The ALM can be obtained by substituting (3) and (6) into (1):

$$y_t = \Phi' z_{t-1} + \eta_t, \qquad (9)$$

where:

$$\Phi' = (\phi_1, \phi'_2, \phi_3) = \left(\alpha + \beta\left(\frac{a_{L,t-1} + a_{H,t-1}}{2} \right), \beta\left(\frac{b'_{L,t-1} + b'_{H,t-1}}{2} \right) + \gamma, \beta\left(\frac{c_{L,t-1}}{2} \right) \right)$$

and $z_{t-1} = \begin{pmatrix} 1 \\ w_{t-1} \\ \hat{y}_{t-1} \end{pmatrix}$.

Back to the Original Questions:

- Is this REE learnable?
- Does the equilibrium change after introducing the interaction assumption?

Expectational Stability

- Recall the E-Stability condition is simply a condition which governs *whether or not a given REE is stable*.

- Evans (1985,1989) devises a condition for mapping the PLM to the ALM – ***Expectational Stability***.

- Evans (1989) defines the E-stability condition in terms of the ordinary differential equation (ODE):

$$\frac{d\theta}{d\tau} = T(\theta) - \theta,$$

where θ is a finite dimensional parameter specified in the PLM, $T(\theta)$ is a mapping (T-mapping) from the PLM to ALM and τ denotes "notional" time.

Least Squares Learning and Recursive Stochastic Algorithms

- ***Agents update their forecasts using Recursive Least Square***

  - Recursive Least Square (RLS) formula:

$$\varphi_{i,t} = \varphi_{i,t-1} + \left(t + T_i\right)^{-1} R_{i,t}^{-1} z_{i,t-1}\left(y_t - \varphi'_{i,t-1} z_{i,t-1}\right)$$

$$R_{i,t} = R_{i,t-1} + \left(t + T_i\right)^{-1}\left(z_{i,t-1} z'_{i,t-1} - R_{i,t-1}\right)$$

for $i \in \{L, H\}$

- To show ***the Convergence of the Estimates***, we need to put the RLS in the form of the Standard ***Stochastic Recursive Algorithm (SRA):***

$$\theta_t = \theta_{t-1} + \gamma_t Q\left(\theta_{t-1}, X_{i,t}, t\right),$$

where

$\theta_t' = \left(vec\left(\varphi_{L,t}\right)', vec\left(\varphi_{H,t}\right)', vec\left(R_{L,t+1}\right), vec\left(R_{H,t+1}\right)\right)$,

$X_{i,t} = \left(z_{i,t}, z_{i,t-1}, \eta_t\right)$

and $\gamma_t = \left(t + T_i\right)^{-1}$.

The Associated Ordinary Differential Equation (ODE) with the SRA:

$$\frac{d\theta}{d\tau} = h(\theta(\tau))$$

where $h(\theta)$ is obtained as:

$h(\theta) = \lim_{t\to\infty} EQ(t, \theta, \bar{X}_t(\theta))$.

$\bar{\theta}$ is an equilibrium point of $\frac{d\theta}{d\tau} = h(\bar{\theta})$ if $h(\bar{\theta}) = 0$.

Lemma 1.

If all eigenvalues of $Dh(\bar{\theta})$ *have negative real parts then* $\bar{\theta}$ *is a locally stable equilibrium point of* $\frac{d\theta}{d\tau} = h(\bar{\theta})$.

If some eigenvalues of $Dh(\overline{\theta})$ *have a positive real part then* $\overline{\theta}$ *is not a locally stable equilibrium point of* $\frac{d\theta}{d\tau} = h(\overline{\theta})$.

Model 1 Illustration

The Rational Expectations Equilibrium:

- Group L:

$$y_t = a_{L,t-1} + b'_{L,t-1} w_{t-1} + c_{L,t-1} \hat{y}_{t-1} + v_t \, ,$$

$$\bar{a}_L = \frac{\alpha}{1-\beta}, \qquad \bar{b}_L = \frac{\gamma}{1-\beta}, \qquad \bar{c}_L = 0$$

- Group H: $y_t = a_{H,t-1} + b'_{H,t-1} w_{t-1} + v_t$,

$$\bar{a}_H = \frac{\alpha}{1-\beta}, \qquad \bar{b}_H = \frac{\gamma}{1-\beta} .$$

E-Stability in Model 1:

From the ordinary differential equation (and the eigenvalues), we have:

Proposition 1. *The system is E-stable if* $\beta < 1$.

Conclusion of Model 1:

1. Assuming both groups have the same (full) information set, Group L discards the expectations of the Group H in the long run (i.e., $\overline{c}_L = 0$).

2. Overparameterization by Group L ***does not change*** the E-stable condition if they learn *independently*.

Model 2 Illustration (Heterogeneous Information Sets)

Main Assumptions:

- Group H has ***full information*** set

$$\left(w_{t-1} \equiv \begin{pmatrix} x_{t-1} \\ w_{2,t-1} \end{pmatrix} \right).$$

- Group L has the ***subset*** of information $\left(x_{t-1} \subset w_{t-1}\right)$.

- Group L learns from Group H with ***error***.

Perceived Law of Motion

Group H's PLM:

$$y_t = a_{H,t-1} + b_{H,t-1}x_{t-1} + c_{H,t-1}w_{2,t-1} + v_t$$

Group L's PLM:

$$y_t = a_{L,t-1} + b_{L,t-1}x_{t-1} + c_{L,t-1}\hat{y}_{t-1} + v_t$$

Restrictive Perceptions Equilibrium (RPE)

- *A Restricted Perceptions equilibrium* is the equilibrium where the forecasts are optimal relative to the restricted information set (Sargent 1991).

Group L's PLM:

$$y_t = a_{L,t-1} + b_{L,t-1} x_{t-1} + c_{L,t-1} \hat{y}_{t-1} + v_t$$

Group H's PLM:

$$y_t = a_{H,t-1} + b_{H,t-1} x_{t-1} + c_{H,t-1} w_{2,t-1} + v_t$$

Group L's RPE:

$$\bar{\varphi}_L = \begin{pmatrix} \bar{a}_L \\ \bar{b}_L \\ \bar{c}_L \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha}{1-\beta}(1-\bar{c}_L) \\ \dfrac{\gamma_1}{1-\beta}(1-\bar{c}_L) \\ \dfrac{(2\gamma_2 + \beta\,\bar{b}_{2H})\,\bar{b}_{2H}\,\sigma^2_{w2}}{(2-\beta)\left(\sigma^2_{\tilde{e}_L} + b^2_{2H}\,\sigma^2_{w2}\right)} \end{pmatrix}$$

Group H's RPE:

$$\bar{\varphi}_H = \begin{pmatrix} \bar{a}_H \\ \bar{b}_{1H} \\ \bar{b}_{2H} \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha}{1-\beta} \\ \dfrac{\gamma_1}{1-\beta} \\ \dfrac{2\gamma_2}{2-\beta(1+\bar{c}_L)} \end{pmatrix}$$

- $\bar{a}_L,\ \bar{b}_{L,}\ \bar{c}_L,\ and\ \bar{b}_{2H}$ ***depend*** on the variance of the measurement error, $\sigma^2_{\tilde{e}_L}$.

Proposition(s): Relating Measurement Error to the RPE's

If $\sigma^2_{\tilde{e}_L} = 0$ then:

- Group L: $\bar{a}_L = 0, \ \ \bar{b}_L = 0, \ \ and \ \ \bar{c}_L = 1$.
- Group H:
  $\bar{a}_H = \dfrac{\alpha}{1-\beta}, \ \ \bar{b}_{1H} = \dfrac{\gamma_1}{1-\beta}, \ \ and \ \ \bar{b}_{2H} = \dfrac{\gamma_2}{1-\beta}$.

If $\sigma^2_{\tilde{e}_L} \rightarrow \infty$, then:

- Group L:

$$\bar{a}_L = \frac{\alpha}{1-\beta}, \quad \bar{b}_L = \frac{\gamma_1}{1-\beta}, \quad and \quad \bar{c}_L = 0 .$$

- Group H:

$$\bar{a}_H = \frac{\alpha}{1-\beta}, \quad \bar{b}_{1H} = \frac{\gamma_1}{1-\beta}, \quad and \quad \bar{b}_{2H} = \frac{2\gamma_2}{2-\beta}.$$

If $\sigma^2_{\tilde{e}_L} \in (0, \infty)$, then

- Group L:

$$\bar{a}_L \in \left(0, \frac{\alpha}{1-\beta}\right), \quad \bar{b}_L \in \left(0, \frac{\gamma_1}{1-\beta}\right), \quad and \quad \bar{c}_L \in (0,1)$$

- Group H:

$$\bar{a}_H = \frac{\alpha}{1-\beta}, \quad \bar{b}_{1H} = \frac{\gamma_1}{1-\beta}, \quad and \quad \bar{b}_{2H} \in \left(\frac{\gamma_2}{1-\beta}, \frac{2\gamma_2}{2-\beta}\right).$$

Expectational Stability in Model 2

- The E-stability condition can be determined by deriving the Jacobian Matrix from the following ordinary differential equation:

$$\frac{d\varphi}{d\tau}=\begin{pmatrix}\alpha+\Theta_{13}\Omega_{43}\gamma_2\\ \gamma_1+\Theta_{23}\Omega_{43}\gamma_2\\ \Theta_{33}\Omega_{43}\gamma_2\\ \alpha\\ \gamma_1\\ \gamma_2\end{pmatrix}+\frac{1}{2}\beta\begin{pmatrix}a_L+a_H+\Theta_{13}\Omega_{43}b_{2H}\\ b_L+b_{1H}+\Theta_{23}\Omega_{43}b_{2H}\\ c_L+\Theta_{33}\Omega_{43}b_{2H}\\ a_L+a_H+a_Hc_L\\ b_L+b_{1H}+b_{1H}c_L\\ b_{2H}+b_{2H}c_L\end{pmatrix}-\begin{pmatrix}a_L\\ b_L\\ c_L\\ a_H\\ b_{1H}\\ b_{2H}\end{pmatrix}$$

Conjecture 1. *Both groups' estimates locally converge to the RPE w.p. 1 if* $\beta<1$.

Boomerang Effect

- ***Boomerang Effect:*** the inaccurate forecasts of Group L confound Group H's convergence to the rational expectations equilibrium and predictive accuracy.

  - **Why Do We See The Boomerang Effect?**

    1. Omitted Variable Problem;

    2. Measurement Error;

    3. Self-referential model

    -- The average expectations of both groups.

The Forecasting Error and the Boomerang Effect

- We use the Mean Squared Error (MSE) to ***measure the forecast accuracy*** for both groups.

MSE for Group L:

$$MSE_L = \begin{cases} \sigma_\eta^2 & if \quad \sigma_{\tilde{e}_L}^2 = 0 \\ \left[\dfrac{2\gamma_2(1-\bar{c}_L)}{2-\beta(1+\bar{c}_L)}\right]^2 \sigma_{w2}^2 + \left[\dfrac{(\beta-2)\bar{c}_L}{2}\right]^2 \sigma_{\tilde{e}_L}^2 + \sigma_\eta^2 & if \quad \sigma_{\tilde{e}_L}^2 \in (0,\infty) \\ \left(\dfrac{2\gamma_2}{2-\beta}\right)^2 \sigma_{w2}^2 + \sigma_\eta^2 & if \quad \sigma_{\tilde{e}_L}^2 \to \infty \end{cases}$$

MSE for Group H:

$$MSE_H = \begin{cases} \sigma_\eta^2 & if \quad \sigma_{\tilde{e}_L}^2 = 0 \ or \ \sigma_{\tilde{e}_L}^2 \to \infty \\ \left[\dfrac{\beta \ \bar{c}_L}{2}\right]^2 \sigma_{\tilde{e}_L}^2 + \sigma_\eta^2 & if \quad \sigma_{\tilde{e}_L}^2 \in (0, \infty) \end{cases}$$

Boomerang Effect on Predictive Accuracy

- When $\sigma^2_{\tilde{e}_L} = 0$, both groups have the ***same*** MSE in the limit $\left(\bar{c}_L = 1\right)$.

- Their MSEs are at the minimum ($MSE_H = MSE_L = \sigma^2_\eta$).

- When $\sigma^2_{\tilde{e}_L} \rightarrow \infty$, Group L eventually ***discards*** the observed expectations from Group H. Both groups learn ***independently***. No boomerang effect occurs.

- When $\sigma^2_{\tilde{e}_L} \in \left(0, \infty\right)$,

    - The presence of measurement error ***leads*** Group L to form an ***incorrect*** expectation via the variable $\hat{y}_{t-1}$;

    - This alters the actual y_t;

Group H fails to forecast the actual y_t correctly.

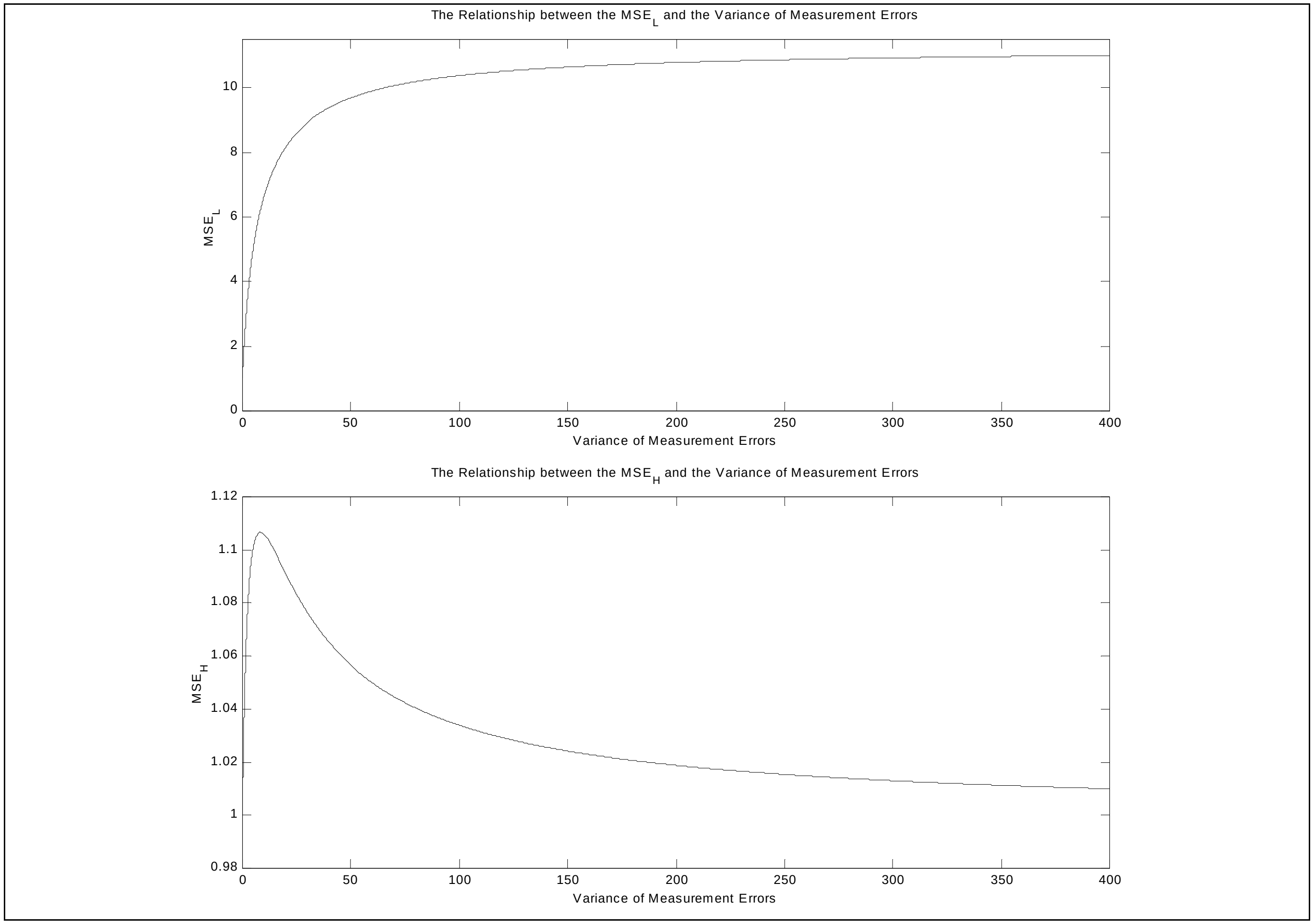

FIGURE 8. MEAN SQUARE ERRORS OF GROUP L AND GROUP H

Boomerang Effect on Coefficients

- Group L's incorrect expectations ***affect*** the value of $\bar{b}_{2H}$ in Group H's expectations.

- Since both groups have common information, x_{t-1}, the boomerang effect ***does not occur*** on the constant coefficient and the coefficient of x_{t-1}.

- But it can ***extend*** to the coefficient of x_{t-1} when $\text{cov}(x_t, w_{2,t}) \neq 0$.

Conclusion and Implications

- The Boomerang Effect exists if there is an interaction among agents in the economy.
- Policy Implications
  - Group H – Monetary Authority
  - Group L – Public
- The inaccurate forecasts of the public may ***affect*** the forecasts of the monetary authority – Boomerang Effect.
- The monetary authority should be more transparent in its actions so that the public is able to obtain more accurate information from it.

Testing the Boomerang Effect

High Group:

$$E^h_{t-1}y_t = a_h + b_{1h}x_{t-1} + b_{2h}w_{2t-1} \tag{1}$$

Low Group:

$$E^l_{t-1}y_t = a_l + b_l x_{t-1} + c_l \hat{y}_{t-1} \tag{2}$$

and

$$\hat{y}_{t-1} \equiv E^h_{t-1}y_t + \tilde{e}_{t-1} \tag{3}$$

Steps of the empirical analysis:

1. Find the sets of common information (x_{t-1}) and advanced information (w_{t-1}).
   - Find the variables that are significant in both group equations as the common information set (x_{t-1}).
   - Pick the variables which are significant in High group but not in Low group as advanced information set (w_{2t-1}).
2. Get the variance of measurement error, $\sigma^2_{\tilde{e}_{t-1}}$:
   - Put equation (3) into equation (2), we get:

$$E^l_{t-1}y_t = a_l + b_l x_{t-1} + c_l \left(E^h_{t-1}y_t + \tilde{e}_{t-1}\right) \tag{4}$$

and

$$E^l_{t-1}y_t = a_l + b_l x_{t-1} + c_l E^h_{t-1}y_t + \xi_t \tag{5}$$

where

$$\xi_t = c_l \tilde{e}_{t-1} \tag{6}$$

therefore, from equations (5) and (6)

$$\tilde{e}_{t-1} = \frac{E^l_{t-1}y_t - \left(a_l + b_l x_{t-1} + c_l E^h_{t-1}y_t\right)}{c_l} \tag{7}$$

3. Rolling-Windows regressions:

$E^l_{t-1}y_t$	x_{t-1}	$E^h_{t-1}y_t$
:	:	:
:	:	:
:	:	:
:	:	:
:	:	:
:	:	:
:	:	:
:	:	:

get $a_l, b_l, c_l \Rightarrow$get $\sigma^2_{\tilde{e}_{t-1}}$ in this window
then roll this window..

4. For each corresponding window, we can also get the MSE for both groups H and L, MSE_H and MSE_L.

5. Finally testing the effect:

$$
\begin{aligned}
MSE_H &= \alpha + \beta\sigma^2_{\tilde{e}_{t-1}} + u \\
MSE_L &= \gamma + \lambda\sigma^2_{\tilde{e}_{t-1}} + \eta
\end{aligned}
$$

According to our theory, we expect that both β and λ are positive – (Boomerang Effect)

6. Robustness Check: alternative the size of windows and information sets...