VI. Module 6: EITM and Simultaneity

A Theoretical Basis for Simultaneity: The Role of Expectations, Learning, and Social Interaction

• Example: The Boomerang Effect

  o Question: What are the consequences for the rational expectations equilibrium if there exists simultaneous interaction among agents in the context of learning dynamics?

  o The inaccurate forecasts by the public (less attentive) may affect the forecasts of the highly attentive (policymakers) – the **Boomerang Effect**.
In previous literature (rational expectations), the expectations are conditioned on all of the information available to the decision makers.

Limitations: Strong Assumption – Not Realistic.

Relax this strong assumption and assume agents are boundedly rational.

In the adaptive learning literature an assumption is typically made that agents have Homogeneous Expectations, heterogeneous expectations, or rational heterogeneous expectations.
What is Missing in the adaptive learning literature?

• All agents are assumed to learn independently. There is no interaction among agents.

• Here we assume interaction among (between) agents.

• Two groups of agents: Less Attentive (Group L) and High Attentive (Group H).

• Two groups are evenly distributed.

• Group L learns from the expectations of Group H with “measurement error”. (Interaction Assumption)
2 Interaction Scenarios:

*Model 1: The Case of Homogeneous Information Sets*

- Both groups of agents are forming their expectations with the *same (full)* information set.

- Results:
  - Group L *relies* on the expectations of Group H for a *limited time*. 
Model 2: The Case of Heterogeneous Information Sets

- Group L has only a subset of relevant information.

- Results:
  
  - Both groups learn the restrictive perceptions equilibrium (RPE).
  
  - Group L finds it necessary to rely on the expectations of Group H.

- Boomerang Effect: the inaccurate forecasts of Group L confound Group H’s convergence to the rational expectations equilibrium and predictive accuracy.
Details of the Model

The True Model (Cobweb Variety):

\[ y_t = \alpha + \beta E_{t-1}^* y_t + \gamma^t w_{t-1} + \eta_t \]  \hspace{1cm} (1)

Notation:

- \( y_t \) = the endogenous variable;
- \( E_{t-1}^* y_t \) = the average expectations of \( y_t \) at time \( t-1 \);
- \( w_{t-1} \) = a vector of exogenous variables.
The Rational Expectations Equilibrium (REE):

\[ y_t = \frac{\alpha}{1 - \beta} + \frac{\gamma'}{1 - \beta} w_{t-1} + \eta_t \]

Questions:

• Is this REE learnable?

• Does the equilibrium change after introducing the interaction assumption?
Examples of Cobweb Applications:

1. Cobweb Model (Muth 1961):

- *Supply-Demand* model in an isolated market.
- Firms produce based on the *expected future price*

\[
d_t = m_I + m_p p_t + v_{1t}
\]
\[
s_t = r_I + r_P p^e_t + r'_w w_{t-1} + v_{2t}
\]

Assuming market clearing, the reduced form of the price function is:

\[
p_t = \mu + \alpha p^e_t + \delta' w_{t-1} + \eta_t
\]
2. Lucas Aggregate Supply Model (Lucas 1973)

\[ q_t = \bar{q} + \pi(p_t - p^e_t) + \varsigma_t \]
\[ m_t + \nu_t = p_t + q_t \]
\[ \nu_t = \mu + \gamma w_{t-1} + \xi_t \]
\[ m_t = \bar{m} + u_t + \rho w_{t-1} \]

The reduced form of the price function is:

\[ p_t = \sum_0 + \sum_1 p^e_t + \sum_2 w_{t-1} + \varepsilon_t \]
Learning Dynamics:

Perceived Law of Motion (PLM)

Group $H$:

\[ y_t = a_{H,t-1} + b'_{H,t-1} w_{t-1} + v_t \]  \hspace{1cm} (2)

Group $H$’s forecast of $y_t$ at time $t-1$:

\[ E_{t-1}^* y_{H,t} = a_{H,t-1} + b'_{H,t-1} w_{t-1} \]  \hspace{1cm} (3)
**Group L:**

\[ y_t = a_{L,t-1} + b'_{L,t-1} w_{t-1} + c_{L,t-1} \hat{y}_{t-1} + v_t \]  \hspace{1cm} (4)

and

\[ \hat{y}_{t-1} = E^*_t y_{H,t} + \tilde{e}_{L,t-1} \]  \hspace{1cm} (5)

where \( \tilde{e}_{L,t-1} \) is the *measurement error* of observing the expectations of Group H.

**Group L’s forecast:**

\[ E^*_{t-1} y_t = a_{L,t-1} + b'_{L,t-1} w_{t-1} + c_{L,t-1} \hat{y}_{t-1} \]  \hspace{1cm} (6)
Actual Law of Motion (ALM)

- *Actual Law of Motion* (ALM) describes the stochastic process of $y_t$ followed by the economy if forecasts are made under the fixed rule given by the PLM.

- Since both groups are *evenly distributed*, the average expectations is:

$$E_{t-1}^* y_t = \frac{E_{t-1}^* y_{H,t} + E_{t-1}^* y_{L,t}}{2}$$  \hspace{1cm} (8)

- Remember, the true model (1) is:

$$y_t = \alpha + \beta E_{t-1}^* y_t + \gamma' w_{t-1} + \eta_t$$

- The ALM can be obtained by substituting (3) and (6) into (1):

$$y_t = \Phi' z_{t-1} + \eta_t,$$  \hspace{1cm} (9)

where:
\[
\Phi' = (\phi_1, \phi'_2, \phi_3) = \left( \alpha + \beta \left( \frac{a_{L,t-1} + a_{H,t-1}}{2} \right), \beta \left( \frac{b'_{L,t-1} + b'_{H,t-1}}{2} \right) + \gamma, \beta \left( \frac{c_{L,t-1}}{2} \right) \right)
\]

and \[ z_{t-1} = \begin{pmatrix} 1 \\ w_{t-1} \\ \hat{y}_{t-1} \end{pmatrix} \]

**Back to the Original Questions:**

- Is this REE learnable?

- Does the equilibrium change after introducing the interaction assumption?
Expectational Stability

• Recall the E-Stability condition is simply a condition which governs whether or not a given REE is stable.

• Evans (1985, 1989) devises a condition for mapping the PLM to the ALM – *Expectational Stability*.

• Evans (1989) defines the E-stability condition in terms of the ordinary differential equation (ODE):

\[
\frac{d\theta}{d\tau} = T(\theta) - \theta,
\]

where $\theta$ is a finite dimensional parameter specified in the PLM, $T(\theta)$ is a mapping (T-mapping) from the PLM to ALM and $\tau$ denotes “notional” time.
Least Squares Learning and Recursive Stochastic Algorithms

- Agents update their forecasts using Recursive Least Square

  - Recursive Least Square (RLS) formula:

    \[
    \varphi_{i,t} = \varphi_{i,t-1} + (t + T_i)^{-1} R_{i,t}^{-1} z_{i,t-1} \left( y_t - \varphi'_{i,t-1} z_{i,t-1} \right)
    \]

    \[
    R_{i,t} = R_{i,t-1} + (t + T_i)^{-1} \left( z_{i,t-1} z'_{i,t-1} - R_{i,t-1} \right)
    \]

    for \( i \in \{L, H\} \)

- To show the Convergence of the Estimates, we need to put the RLS in the form of the Standard Stochastic Recursive Algorithm (SRA):

  \[
  \theta_t = \theta_{t-1} + \gamma_t Q(\theta_{t-1}, X_{i,t}, t)
  \]

  where

  \[
  \theta'_t = \text{vec}(\varphi_{L,t}), \text{vec}(\varphi_{H,t}), \text{vec}(R_{L,t+1}), \text{vec}(R_{H,t+1})
  \]

  \[
  X_{i,t} = (z_{i,t}, z_{i,t-1}, \eta_t)
  \]

  and \( \gamma_t = (t + T_i)^{-1} \).
The Associated Ordinary Differential Equation (ODE) with the SRA:

\[ \frac{d\theta}{d\tau} = h(\theta(\tau)) \]

where \( h(\theta) \) is obtained as:

\[ h(\theta) = \lim_{t \to \infty} EQ(t, \theta, \bar{X}_t(\theta)) \].

\( \bar{\theta} \) is an equilibrium point of \( \frac{d\theta}{d\tau} = h(\bar{\theta}) \) if \( h(\bar{\theta}) = 0 \).

Lemma 1.

If all eigenvalues of \( Dh(\bar{\theta}) \) have negative real parts then \( \bar{\theta} \) is a locally stable equilibrium point of

\[ \frac{d\theta}{d\tau} = h(\bar{\theta}) \].
If some eigenvalues of $Dh(\overline{\theta})$ have a positive real part then $\overline{\theta}$ is not a locally stable equilibrium point of $\frac{d\theta}{d\tau} = h(\overline{\theta})$. 
Model 1 Illustration

The Rational Expectations Equilibrium:

- Group L:

\[ y_t = a_{L,t-1} + b'_{L,t-1} w_{t-1} + c_{L,t-1} \hat{y}_{t-1} + v_t, \]

\[ \bar{a}_L = \frac{\alpha}{1 - \beta}, \quad \bar{b}_L = \frac{\gamma}{1 - \beta}, \quad \bar{c}_L = 0 \]

- Group H:

\[ y_t = a_{H,t-1} + b'_{H,t-1} w_{t-1} + v_t, \]

\[ \bar{a}_H = \frac{\alpha}{1 - \beta}, \quad \bar{b}_H = \frac{\gamma}{1 - \beta}. \]
E-Stability in Model 1:

From the ordinary differential equation (and the eigenvalues), we have:

Proposition 1. The system is E-stable if $\beta < 1$.

Conclusion of Model 1:

1. Assuming both groups have the same (full) information set, Group L discards the expectations of the Group H in the long run (i.e., $\overline{c}_L = 0$).

2. Overparameterization by Group L does not change the E-stable condition if they learn independently.
Model 2 Illustration (Heterogeneous Information Sets)

Main Assumptions:

- Group H has **full information** set

\[
\begin{pmatrix}
  w_{t-1} \\
  x_{t-1} \\
  w_{2,t-1}
\end{pmatrix}.
\]

- Group L has the **subset** of information

  \[
  (x_{t-1} \subset w_{t-1}).
  \]

- Group L learns from Group H with **error**.
Perceived Law of Motion

*Group H’s PLM:*

\[ y_t = a_{H,t-1} + b_{H,t-1} x_{t-1} + c_{H,t-1} w_{2,t-1} + v_t \]

*Group L’s PLM:*

\[ y_t = a_{L,t-1} + b_{L,t-1} x_{t-1} + c_{L,t-1} \hat{y}_{t-1} + v_t \]
Restrictive Perceptions Equilibrium (RPE)

- *A Restricted Perceptions equilibrium* is the equilibrium where the forecasts are optimal relative to the restricted information set (Sargent 1991).

**Group L’s PLM:**

\[ y_t = a_{L,t-1} + b_{L,t-1} x_{t-1} + c_{L,t-1} \hat{y}_{t-1} + v_t \]

**Group H’s PLM:**

\[ y_t = a_{H,t-1} + b_{H,t-1} x_{t-1} + c_{H,t-1} w_{2,t-1} + v_t \]
Group L’s RPE:

\[
\bar{\varphi}_L = \begin{pmatrix}
\bar{a}_L \\
\bar{b}_L \\
\bar{c}_L
\end{pmatrix} = \begin{pmatrix}
\frac{\alpha}{1 - \beta} (1 - \bar{c}_L) \\
\frac{\gamma_1}{1 - \beta} (1 - \bar{c}_L) \\
(2 \gamma_2 + \beta \bar{b}_2) \bar{b}_2 \sigma_{w2}^2 \\
(2 - \beta) (\sigma_{\varepsilon L}^2 + \bar{b}_2^2 \sigma_{w2}^2)
\end{pmatrix}
\]

Group H’s RPE:

\[
\bar{\varphi}_H = \begin{pmatrix}
\bar{a}_H \\
\bar{b}_H \\
\bar{b}_{2H}
\end{pmatrix} = \begin{pmatrix}
\frac{\alpha}{1 - \beta} \\
\frac{\gamma_1}{1 - \beta} \\
2 \gamma_2 \\
2 - \beta (1 + \bar{c}_L)
\end{pmatrix}
\]

- \( \bar{a}_L, \bar{b}_L, \bar{c}_L, \) and \( \bar{b}_{2H} \) depend on the variance of the measurement error, \( \sigma_{\varepsilon L}^2 \).
Proposition(s): Relating Measurement Error to the RPE’s

If $\sigma_{\tilde{e}_L}^2 = 0$ then:

- Group L: $\bar{a}_L = 0,$ $\bar{b}_L = 0,$ and $\bar{c}_L = 1.$

- Group H:

$$\bar{a}_H = \frac{\alpha}{1 - \beta}, \quad \bar{b}_{1H} = \frac{\gamma_1}{1 - \beta}, \quad \text{and} \quad \bar{b}_{2H} = \frac{\gamma_2}{1 - \beta}.$$
If $\sigma_{\tilde{e}_L}^2 \to \infty$, then:

- **Group L:**

  \[
  \bar{a}_L = \frac{\alpha}{1-\beta}, \quad \bar{b}_L = \frac{\gamma_1}{1-\beta}, \text{ and } \bar{c}_L = 0.
  \]

- **Group H:**

  \[
  \bar{a}_H = \frac{\alpha}{1-\beta}, \quad \bar{b}_{1H} = \frac{\gamma_1}{1-\beta}, \text{ and } \bar{b}_{2H} = \frac{2\gamma_2}{2-\beta}.
  \]
If $\sigma_{\tilde{e}_L}^2 \in (0, \infty)$, then

- Group L:

$$\overline{a}_L \in \left(0, \frac{\alpha}{1-\beta}\right), \quad \overline{b}_L \in \left(0, \frac{\gamma_1}{1-\beta}\right), \quad \text{and} \quad \overline{c}_L \in (0,1)$$

- Group H:

$$\overline{a}_H = \frac{\alpha}{1-\beta}, \quad \overline{b}_1 = \frac{\gamma_1}{1-\beta}, \quad \text{and} \quad \overline{b}_2 \in \left(\frac{\gamma_2}{1-\beta}, \frac{2\gamma_2}{2-\beta}\right)$$
Expectational Stability in Model 2

- The E-stability condition can be determined by deriving the Jacobian Matrix from the following ordinary differential equation:

\[
\frac{d\varphi}{d\tau} = \begin{pmatrix}
\alpha + \Theta_{13}\Omega_{43}\gamma_2 \\
\gamma_1 + \Theta_{23}\Omega_{43}\gamma_2 \\
\Theta_{33}\Omega_{43}\gamma_2
\end{pmatrix} + \frac{1}{2} \beta
\begin{pmatrix}
\Theta_{13}\Omega_{43}b_{2H} \\
\Theta_{23}\Omega_{43}b_{2H} \\
\Theta_{33}\Omega_{43}b_{2H}
\end{pmatrix}
\begin{pmatrix}
a_L + a_H + a_Hc_L \\
b_L + b_{1H} + b_{1H}c_L \\
b_{2H} + b_{2H}c_L
\end{pmatrix}
\begin{pmatrix}
a_L \\
b_L \\
c_L \\
a_H \\
b_{1H} \\
b_{2H}
\end{pmatrix}
\]

Conjecture 1. Both groups’ estimates locally converge to the RPE w.p. 1 if \( \beta < 1 \).
Boomerang Effect

- **Boomerang Effect**: the inaccurate forecasts of Group L confound Group H’s convergence to the rational expectations equilibrium and predictive accuracy.

- **Why Do We See The Boomerang Effect?**
  1. Omitted Variable Problem;
  2. Measurement Error;
  3. Self-referential model

-- The average expectations of both groups.
The Forecasting Error and the Boomerang Effect

- We use the Mean Squared Error (MSE) to measure the forecast accuracy for both groups.

MSE for Group L:

\[
MSE_L = \begin{cases} 
\frac{2\gamma_2(1 - \bar{c}_L)}{2 - \beta(1 + \bar{c}_L)} \sigma_{w_2}^2 + \left[ \frac{(\beta - 2)\bar{c}_L}{2} \right] \sigma_{\varepsilon_L}^2 + \sigma_\eta^2 & \text{if } \sigma_{\varepsilon_L}^2 = 0 \\
\left( \frac{2\gamma_2}{2 - \beta} \right)^2 \sigma_{w_2}^2 + \sigma_\eta^2 & \text{if } \sigma_{\varepsilon_L}^2 \in (0, \infty) \\
\rightarrow \infty & \text{if } \sigma_{\varepsilon_L}^2 \rightarrow \infty
\end{cases}
\]
MSE for Group H:

\[
MSE_H = \begin{cases} 
\sigma^2_{\eta} & \text{if } \sigma^2_{\tilde{e}_L} = 0 \text{ or } \sigma^2_{\tilde{e}_L} \to \infty \\
\left(\frac{\beta \bar{c}_L}{2}\right)^2 \sigma^2_{\tilde{e}_L} + \sigma^2_{\eta} & \text{if } \sigma^2_{\tilde{e}_L} \in (0, \infty)
\end{cases}
\]
Boomerang Effect on Predictive Accuracy

• When $\sigma_{\tilde{e}_L}^2 = 0$, both groups have the same MSE in the limit $\left(\bar{c}_L = 1\right)$.

• Their MSEs are at the minimum $(MSE_H = MSE_L = \sigma_\eta^2)$.

• When $\sigma_{\tilde{e}_L}^2 \rightarrow \infty$, Group L eventually discards the observed expectations from Group H. Both groups learn independently. No boomerang effect occurs.

• When $\sigma_{\tilde{e}_L}^2 \in (0, \infty)$,
  
  o The presence of measurement error leads Group L to form an incorrect expectation via the variable $\hat{Y}_{t-1}$;

  o This alters the actual $Y_t$;

Group H fails to forecast the actual $Y_t$ correctly.
Figure 8. Mean Square Errors of Group L and Group H
Boomerang Effect on Coefficients

- Group L’s incorrect expectations affect the value of $\bar{b}_{2H}$ in Group H’s expectations.

- Since both groups have common information, $x_{t-1}$, the boomerang effect does not occur on the constant coefficient and the coefficient of $x_{t-1}$.

- But it can extend to the coefficient of $x_{t-1}$ when $\text{cov}(x_t, w_{2,t}) \neq 0$. 
Conclusion and Implications

• The Boomerang Effect exists if there is an interaction among agents in the economy.

• Policy Implications
  
  o Group H – Monetary Authority
  
  o Group L – Public

• The inaccurate forecasts of the public may affect the forecasts of the monetary authority – Boomerang Effect.

• The monetary authority should be more transparent in its actions so that the public is able to obtain more accurate information from it.
Testing the Boomerang Effect

High Group:

\[ E_{t-1}^h y_t = a_h + b_1 x_{t-1} + b_2 w_{2t-1} \]  \hspace{1cm} (1)

Low Group:

\[ E_{t-1}^l y_t = a_l + b_l x_{t-1} + c_l \hat{y}_{t-1} \]  \hspace{1cm} (2)

and

\[ \hat{y}_{t-1} = E_{t-1}^h y_t + \hat{e}_{t-1} \]  \hspace{1cm} (3)

Steps of the empirical analysis:

1. Find the sets of common information \((x_{t-1})\) and advanced information \((w_{1t-1})\).
   
   - Find the variables that are significant in both group equations as the common information set \((x_{t-1})\).
   
   - Pick the variables which are significant in High group but not in Low group as advanced information set \((w_{2t-1})\).

2. Get the variance of measurement error, \(\sigma_{\hat{e}_{t-1}}^2\):
   
   - Put equation (3) into equation (2), we get:
     \[ E_{t-1}^l y_t = a_l + b_l x_{t-1} + c_l \left( E_{t-1}^h y_t + \hat{e}_{t-1} \right) \]  \hspace{1cm} (4)

   and

   \[ E_{t-1}^l y_t = a_l + b_l x_{t-1} + c_l E_{t-1}^h y_t + \xi_t \]  \hspace{1cm} (5)

   where

   \[ \xi_t = c_l \hat{e}_{t-1} \]  \hspace{1cm} (6)

   therefore, from equations (5) and (6)

   \[ \hat{e}_{t-1} = \frac{E_{t-1}^l y_t - \left( a_l + b_l x_{t-1} + c_l E_{t-1}^h y_t \right)}{c_l} \]  \hspace{1cm} (7)
3. Rolling-Windows regressions:

\[
E_{t-1} y_t x_{t-1} E_{t-1}^h y_t
\]

get \( a_t, b_t, c_t \Rightarrow \sigma^2_{\epsilon_{t-1}} \) in this window

then roll this window..

4. For each corresponding window, we can also get the MSE for both groups H and L, \( MSE_H \) and \( MSE_L \).

5. Finally testing the effect:

\[
MSE_H = \alpha + \beta \sigma^2_{\epsilon_{t-1}} + u \\
MSE_L = \gamma + \lambda \sigma^2_{\epsilon_{t-1}} + \eta
\]

According to our theory, we expect that both \( \beta \) and \( \lambda \) are positive — (Boomerang Effect)

6. Robustness Check: alternative the size of windows and information sets...