The Boomerang Effect: Learning From the Expectations of Others

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In this paper, we consider a framework with two groups of agents. Both groups use least squares learning to make their forecasts. However, both groups either have the same or heterogeneous information sets and the less attentive group (Group L) uses the expectations of the more, or highly, attentive group (Group H) to update their forecasts. The E-stability condition remains unchanged. In the case of heterogeneous information, the inaccuracy of Group L's forecasts leads to what we call the boomerang effect: the inaccurate forecasts of Group L now confound and delay Group H's convergence to the rational expectations equilibrium. (*JEL* D83, D84, E17)

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* Earlier versions of this paper were presented at the Annual Meeting of the American Political Science Association, San Francisco, California (August 30-September 2, 2001). Granato is currently Political Science Program Director, National Science Foundation, 4201 Wilson Blvd., Suite 980, Arlington, Virginia 22239 (e-mail: jgranato@nsf.gov). Wong is in the Department of Economics, Finance and International Business, University of Southern Mississippi, Box 5072, Hattiesburg, Mississippi 39406-5072 (e-mail: wong@cba.usm.edu; phone: 601-266-5929). We would like to thank Mary Bange, Shankha Chakraborty, George Evans, Andreas Flache, Eran Guse, Stephen Haynes, Mark Jones, Sandra Sizer Moore, Mike Munger, Jim Robinson, and Kevin Terada for their comments and insights. The views and findings in this paper are those of the authors and do not necessarily reflect those of the National Science Foundation.

I. Introduction

How can decentralized political and economic systems, with uneven agent information levels, exhibit coherent behavior? Considerable research, debate, and even some common sense, suggest it is fantasy. A chief obstacle is that agents lack the information necessary for accurate predictions. After all, good forecasts require useful information. And while agents do have specific knowledge of the political and economic information in their immediate setting this familiarity is confounded by their imperfect knowledge of their more general surroundings.

On its face, coordinating these two types of information — specific and general — is a formidable challenge. Yet, there are ways to minimize this knowledge and forecast deficit. One way is through formal and informal social interaction where agents, who pay more attention and have better information, pass on their information and forecasts to less attentive agents. In the limit, after a good deal of learning occurs, the less attentive agents can forecast as if they are the highly attentive agents. Accurate articulation on political and economic matters, through increasingly accurate predictions by all agents, is achieved with the passage of time.

In the past, researchers have dealt with forecasting and expectation issues by making heroic assumptions about agent's abilities. Rational expectations forecasts assume forecasts are conditioned on all information available to decision makers [Lucas, 1972, 1973; Sargent, 1973]. Since agents are able to obtain all information available, their forecasts are always the most efficient. More recently, attention has shifted to less demanding informational capabilities and the process by which agents can adjust and attain rational expectations forecasts [Evans and Honkapohja, 2001].

The adjustment process where the exchange of information produces equilibrium forecasts is the subject of this paper. We apply the cobweb model to an adaptive learning process to allow for heterogeneous information and attention levels. This analysis examines how agents' forecasts are influenced by the forecasts of others [Durlauf and Young, 2001]. The primary focus of this prior research is on the diffusion of information. The social interaction is thought to be asymmetric: a more attentive group's forecasts and information influence the less attentive group's forecasts [Granato and Krause, 2000; Romer and Romer, 2000].

The cobweb model is particularly well suited for this analysis since its wide usage in macroeconomic studies points to the generalizability of our findings [See Ezekiel, 1938; Muth, 1961; Arifovic, 1994; Brock and Hommes, 1997; Evans and Honkapohja, 2001; Branch, 2002]. The model can be considered a reduced form of supply and demand in an isolated market. For example, consider a single competitive market where a time lag exists in production. Firms are price takers who set the level of production according to their price expectations before they observe the actual price level. Market equilibrium of price and quantity is determined by firms' expected profit maximization and exogenously given demand. Alternatively, Lucas [1973] derives the reduced form of inflation behavior which is consistent with the cobweb model.

There has also been extensive study of the cobweb model's stability properties. Ezekiel [1938] first argued that the cobweb theorem is consistent with the convergence of the market equilibrium price provided the ratio of the slopes of demand and supply is less than one. Recent studies incorporate the rational expectations assumption to examine the stability of rational expectations equilibrium (REE) in a dynamic cobweb model [Brock and Hommes, 1997; Branch, 2002]. Further, Evans and Honkapohja [2001] explore the stability condition of the cobweb model in the context of adaptive learning.

We extend this research to the properties of the equilibrium in the cobweb model using the adaptive learning process with interactive agents. This interaction involves one group that is less attentive and informed learning from another group that is more attentive and informed. Based on this interaction we introduce “the boomerang effect,” which we define as a situation in which the inaccurate forecasts of a less attentive group confound a more attentive group’s forecasts. In other words, we explore the situation where even when one group of agents has full information to make forecasts, they cannot obtain the REE due to the interaction with another group that has far less information and less accurate forecasting abilities. Not only is the REE unobtainable, but the forecast errors of these attentive and well informed agents become larger.

The specifics of our model are as follows. We call one group, Group L, because agents in this group are less attentive to the issues of the day. Group L interacts with a more attentive group, Group H. We assume that over the course of this interaction, Group L incorporates the expectations of the more attentive Group H.[1] However, Group L is not able to receive a certain part of the information set. In forming their expectations, they can only use a subset of information and thus have a stronger need to make use of the expectations from Group H. The result is that Group L relies on the expectations of Group H, even asymptotically.

Despite this dependence between Group L and Group H, the relative weight (or reliance) that Group L places on the expectations of Group H depends crucially on the size of Group L’s observational error of Group H’s expectations. This situation describes the results in a restricted perceptions equilibrium (RPE), which Evans and Honkapohja [2001, p. 320] define as an optimal forecast in the face of a restricted information set (used by agents) [Marcet and Sargent, 1989b and Sargent, 1991].

The accuracy of this information is also important for two other reasons. The equilibrium parameters depend on how precisely Group L observes the expectations of Group H. It is the size of this observation error variance that creates what we call the boomerang effect on the accuracy of Group H's forecasts. We further demonstrate how, under a stability condition, least squares learning converges to an RPE. We also show that learning from others' expectations does not affect the stability under which learning converges to the restrictive perceptions equilibrium. We use simulations to illustrate our results.

Important policy implications also follow from interaction between the public and policy-makers. If the public observes policymaker information with error and then uses this information to make forecasts, then the boomerang effect emerges. As a result of the boomerang effect, high information groups such as policymakers will fail to make accurate forecasts of the economic conditions because of the feedback from the public. Consequently, the boomerang effect on policy causes more uncertainty about a macroeconomic target with the eventual result being failed stabilization policies and additional economic volatility.

Indeed, recent research on learning and monetary policy focuses on situations where agents hold possibly different expectations than the monetary authority [Romer and Romer, 2000]. The boomerang effect is relevant for this line of research because it points to issues of transparency when monetary authorities make policy statements or take policy actions. The model in this paper suggests a formal framework on the effects of policy pronouncements that are aimed at shaping private sector expectations.

This paper is organized as follows. Section II provides the basic description of the cobweb model. Section III then extends the cobweb model to include interaction between agents in Group L and Group H. This interaction includes Group L, borrowing the expectations (with errors) from the more attentive group, Group H. We use simulations to present results. The results center on the conditions for convergence to equilibrium. It is shown that the equilibrium depends on the information structure which can be learned under a suitable stability condition. In Section

IV we demonstrate the boomerang effect where the properties of the equilibrium include how the degree of Group L's observation errors affects the convergence of Group H's equilibrium forecasts. Section V discusses the boomerang effect results and Section VI concludes.

II. The Cobweb Model: Applications

Arifovic [1994] and Evans and Honkapohja [2001] presents a simple cobweb model where there are n firms in a competitive market that produce a homogeneous product. Each individual firm i chooses the optimal individual quantity, $q_{i,t}^{s}$, to maximize its expected profit according to its (rational or nonrational) expectation of p_t formed at the end of time $t-1$, (i.e., $E_{t-1}^{*}p_t$). Aggregate supply, $Q_t^s = \sum_{i=1}^{n} q_{i,t}^{s}$, is given as:

$$Q_t^s = \chi_0 + \chi_1 E_{t-1}^{*} p_t + \chi_2' w_{t-1} + \upsilon_{st},$$

where $\chi_1 > 0$, χ_0 is an intercept, w_{t-1} is a vector of production determinants and υ_{st} are white noise supply shocks.

The market price p_t which clears the market at time t is determined by market demand:

$$Q_t^d = \vartheta_0 - \vartheta_1 p_t + \upsilon_{dt},$$

where $\vartheta_1 > 0$, ϑ_0 is an intercept, and υ_{dt} are white noise demand shocks.

In equilibrium ($Q_t^d = Q_t^s$), the reduced-form of the model is:

$$p_t = \alpha + \beta E_{t-1}^{*} p_t + \gamma' w_{t-1} + \eta_t, \tag{1}$$

where $\alpha \equiv (\vartheta_0 - \chi_0)/\vartheta_1, \beta \equiv -\chi_1/\vartheta_1 < 0$, and $\gamma' = -\chi_2'/\vartheta_1$. $\eta_t = \left(Q_t^d - Q_t^s\right)/\vartheta_1$, so that $\eta_t \sim \text{iid}\left(0, \sigma_\eta^2\right)$. In equation (1), the market price (p_t) is determined by its expectation $\left(E_{t-1}^{*} p_t\right)$ and other observable factors (w_{t-1}) and stochastic shocks (η_t).

A cobweb model (1) is particularly useful for our purposes.[2] Different versions of cobweb model have also been studied in the learning literature. The reason is that the cobweb model has some special characteristics. It is linear univariate and it can possess a unique REE. The cobweb model also has an added feature that is also used in the learning literature. It is a self-referential model where the endogenous variable(s) depend on forecasts (expectations) of it. The forecasts from self-referential models may or may not contain the REE. This attribute allows for the possibility that agents learn the REE.

III. Interactive Agents in the Cobweb Model

Our model contains two groups of people. The first group, Group L, is less attentive. These agents tend to be not as up-to-date on political and economic events or institutions. Members of the second group, Group H, can be characterized as opinion leaders who are highly attentive and up-to-date on political and economic events and institutions. These opinion leaders, what political scientists and sociologists call issue publics are key in any diffusion of information, since they are a source of the information. For simplicity, we distribute these two groups evenly in the population.

We assume that both groups operate under the cobweb model (1) which represents the data generation process (DGP) or true model:

$$y_t = \alpha + \beta E^*_{t-1} y_t + \gamma' w_{t-1} + \eta_t, \tag{2}$$

where y_t is an endogenous variable, $E^*_{t-1} y_t$ is the (rational or nonrational) expectation of y_t formed at the end of time $t-1$, w_{t-1} is a $p \times 1$ vector of exogenous observables following a stationary VAR (without loss of generality, we assume w_t has zero mean for convenience), α is a constant, β is a parameter, $\gamma'$is a $1 \times$ p vector, and η_t is $iid(0, \sigma^2_\eta)$.

III.A. The Perceived Law of Motion and Actual Law of Motion[3]

We assume that agents act like econometricians and forecast y_t by running least squares regressions of y_t on past information w_{t-1}. If agents learn independently with the same observable information,[4] then the forecast for each group is based on a perceived law of motion (PLM) of the form:

$$y_t = a_{i,t-1} + b'_{i,t-1} w_{t-1} + v_t, \tag{3}$$

where $i \in \{L, H\}$, $E(v_t) = 0$, a_i is a constant, and b'_i is a $1 \times p$ vector. Subscripts L and H represent Group L and Group H, respectively.

We assume that both groups possess different information sets $(x_{t-1,} w_{t-1})$.[5] For simplicity, w_t is a 2×1 vector of exogenous observations and x_t is a 1×1 vector of exogenous observations where $x_t \subset w_t$. The highly attentive group has the full information set (w_{t-1}) and the less attentive group has a subset of information (x_{t-1}). Moreover, We assume agents are interactive that Group L learns from Group H so that their PLM is:

$$y_t = a_{L,t-1} + b_{L,t-1} x_{t-1} + c_{L,t-1} \widehat{y}_{t-1} + v_t \tag{4}$$

and

$$\widehat{y}_{t-1} = E^*_{H,t-1} y_t + \tilde{e}_{L,t-1}, \tag{5}$$

where $\tilde{e}_{L,t-1} \sim iid\left(0, \sigma^2_{\tilde{e}_L}\right)$ and $\widehat{y}_{t-1}$is the observed information that Group L, the less attentive group, gets from Group H, $E^*_{H,t-1} y_t$ (See equation (8)), with observational error $(\tilde{e}_{L,t-1})$ at time $t-1$. Since Group L obtains the observed information after Group H forms its expectations, Group L treats the observed information as a predetermined variable.

We add the error term $(\tilde{e}_L)$ for two reasons. First, because incorporating the error term reduces the collinearity between the variables. Second, even though collinearity exists, Group L agents cannot have the exact data from Group H agents. As a result, when the less attentive agents forecast and update their forecast (4), they generate larger standard errors on the common parameters.

The highly attentive group, Group H, possesses the full information to forecast the variable of interest. Since $x_{t-1} \subset w_{t-1}$, we can partition the information set w_{t-1} into two parts: $w_{t-1} \equiv \begin{pmatrix} x_{t-1} \\ w_{2,t-1} \end{pmatrix}$ and $b'_{H,t-1} \equiv \begin{pmatrix} b_{1H,t-1} \\ b_{2H,t-1} \end{pmatrix}$, where $w_{2,t}$ is a 1×1 vector of exogenous observations. We call this vector advanced information. To simplify the analytical calculations we assume that the covariance between x_t and $w_{2,t}$ is zero ($cov\left(x_t, w_{2,t}\right) = 0$). We also provide numerical results in the case that $cov\left(x_t, w_{2,t}\right) \neq 0$. Therefore, the PLM for Group H is:

$$y_t = a_{H,t-1} + b_{1H,t-1}x_{t-1} + b_{2H,t-1}w_{2,t-1} + v_t. \tag{6}$$

Now suppose both groups forecast following the process of equations (4) and (6), respectively, and they have data on the political economic system from periods $t_i = T_i, ..., t-1$, where $i \in \{L, H\}$. Therefore, the time $t-1$ information set for the less attentive group, Group L, is $\{y_i, x_i, \widehat{y}_i\}_{i=T_L}^{t-1}$ but the information set for the highly attentive group, Group H, at time $t-1$ is $\{y_i, w_i\}_{i=T_H}^{t-1}$. The two groups use (7) and (8), respectively, to forecast the variable of interest:[6]

$$E^*_{L,t-1}y_t = \varphi'_{L,t-1}z_{L,t-1} \tag{7}$$

and

$$E^*_{H,t-1}y_t = \varphi'_{H,t-1}z_{H,t-1}, \tag{8}$$

where $z'_{L,t-1} \equiv (1, x_{t-1}, \widehat{y}_{t-1})$, $z'_{H,t-1} \equiv (1, x_{t-1}, w_{2,t-1})$, $\varphi'_{L,t-1} \equiv (a_{L,t-1}, b_{L,t-1}, c_{L,t-1})$ and $\varphi'_{H,t-1} \equiv (a_{H,t-1}, b_{1H,t-1}, b_{2H,t-1})$.

Since we assume that both groups are evenly distributed, the average expectation $E^*_{t-1}y_t$ is:

$$E^*_{t-1}y_t = \frac{E^*_{L,t-1}y_t + E^*_{H,t-1}y_t}{2}. \tag{9}$$

At this point we can use the true model (2). We can obtain the actual law of motion (ALM) by substituting (7) and (8) into (2), yielding:

$$y_t = \Phi' z_{t-1} + \eta_t, \tag{10}$$

where $\Phi' \equiv (\phi_1, \phi_2, \phi_3, \phi_4) = (\alpha + \beta\left(\frac{a_{L,t-1}+a_{H,t-1}}{2}\right), \beta\left(\frac{b_{L,t-1}+b_{1H,t-1}}{2}\right) + \gamma_1, \beta\left(\frac{b_{2H,t-1}}{2}\right) +$

$\gamma_2, \beta\left(\frac{c_{L,t-1}}{2}\right)\right), z_{t-1} = (1, x_{t-1}, w_{2,t-1}, \widehat{y}_{t-1})$. The data are partitioned, $\gamma \equiv (\gamma_1, \gamma_2)$.

III.B. Restrictive Perceptions Equilibrium

In this case, one group has a restricted information set, but this does not preclude the possibility for optimal forecasting. We establish that an RPE exists in this section. We first solve the ordinary differential equation (ODE) using equations (2), (4), (6) and (18) as follows:

$$\frac{d\varphi}{d\tau} = \begin{pmatrix} \alpha + \Pi_1\gamma_2 \\ \gamma_1 + \Pi_2\gamma_2 \\ \Pi_3\gamma_2 \\ \alpha \\ \gamma_1 \\ \gamma_2 \end{pmatrix} + \frac{1}{2}\beta \begin{pmatrix} a_L + a_H + \Pi_1 b_{2H} \\ b_L + b_{1H} + \Pi_2 b_{2H} \\ c_L + \Pi_3 b_{2H} \\ a_L + a_H + a_H c_L \\ b_L + b_{1H} + b_{1H} c_L \\ b_{2H} + b_{2H} c_L \end{pmatrix} - \begin{pmatrix} a_L \\ b_L \\ c_L \\ a_H \\ b_{1H} \\ b_{2H} \end{pmatrix}, \tag{11}$$

where $\varphi' \equiv (a_L, b_L, c_L, a_H, b_{1H}, b_{2H})$, $\Pi_1 = -\frac{a_H b_{2H} \sigma_{w2}^2}{\sigma_{\tilde{e}_L}^2 + b_{2H}^2 \sigma_{w2}^2}$, $\Pi_2 = -\frac{b_{1H}(b_{2H}\sigma_{w2}^2)}{\sigma_{\tilde{e}_L}^2 + b_{2H}^2 \sigma_{w2}^2}$, and $\Pi_3 = \frac{b_{2H}\sigma_{w2}^2}{\sigma_{\tilde{e}_L}^2 + b_{2H}^2 \sigma_{w2}^2}$.

One can solve for the RPE by setting the ODE equal to zero. We have:

$$\bar{\varphi}_L = \begin{pmatrix} \bar{a}_L \\ \bar{b}_L \\ \bar{c}_L \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{1-\beta}(1 - \bar{c}_L) \\ \frac{\gamma_1}{1-\beta}(1 - \bar{c}_L) \\ \frac{(2\gamma_2 + \beta \bar{b}_{2H})\bar{b}_{2H}\sigma_{w2}^2}{(2-\beta)(\sigma_{\tilde{e}_L}^2 + \bar{b}_{2H}^2 \sigma_{w2}^2)} \end{pmatrix} \tag{12}$$

and

$$\bar{\varphi}_H = \begin{pmatrix} \bar{a}_H \\ \bar{b}_{1H} \\ \bar{b}_{2H} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{1-\beta} \\ \frac{\gamma_1}{1-\beta} \\ \frac{2\gamma_2}{2-\beta(1+\bar{c}_L)} \end{pmatrix}. \tag{13}$$

The observational error $\tilde{e}_{L,t-1}$ plays a very important role in the model. Whether the less attentive group uses the observed information from the highly attentive group depends on how accurately Group L obtains information (the expectations) from Group H.

The accuracy is represented by the variance of the observational error, $\sigma_{\tilde{e}_L}^2$. Equation (12) implies that $0 \leq \bar{c}_L \leq 1$. If the less attentive group can accurately obtain the highly attentive group's expectations (i.e., $\sigma_{\tilde{e}_L}^2 \to 0$), then we can see that $\bar{c}_L = 1$ by solving equations (12) and (13) with $\sigma_{\tilde{e}_L}^2 = 0$. In addition, $\bar{c}_L \to 0$ as $\sigma_{\tilde{e}_L}^2 \to \infty$. Moreover, the values of $\bar{c}_L$ affect $\bar{a}_L$ and $\bar{b}_L$. If $\bar{c}_L \to 0$, $\bar{a}_L \to \frac{\alpha}{1-\beta}$ and $\bar{b}_L \to \frac{\gamma_1}{1-\beta}$, and both $\bar{a}_L$, $\bar{b}_L \to 0$ if $\bar{c}_L \to 1$. However, under the assumption that the covariance between x_t and $w_{2,t}$ is zero, $\bar{c}_L$ does not affect $\bar{a}_H$ and $\bar{b}_{1H}$ at all.

Both will approach the REE, $(\bar{a}_H, \bar{b}_{1H}) \rightarrow \left(\frac{\alpha}{1-\beta}, \frac{\gamma_1}{1-\beta}\right)$.[7]

Intuitively, if $\sigma^2_{\tilde{e}_L}$ is close to zero, Group L can obtain very accurate information from the highly attentive group. Therefore, Group L uses Group H's information to form their expectations. Group L also deemphasizes, in the limit, their initial information. Conversely, if $\sigma^2_{\tilde{e}_L}$ is sufficiently large $\left(\sigma^2_{\tilde{e}_L} \rightarrow \infty\right)$, then the less attentive group is no longer able to predict accurately the highly attentive group's expectations. In this situation, the less attentive group treats $\widehat{y}_{t-1}$ as a noise term only. Group L thereby puts less weight on the highly attentive group's expectations and more emphasis on their own information.

Finally, equation (13) shows that $\bar{b}_{2H}$ is affected by $\bar{c}_L$ ranging between $\left(\frac{\gamma_2}{1-\beta}, \frac{2\gamma_2}{2-\beta}\right)$. This latter relation is evidence of what we call *the boomerang effect on the RPE* (Proposition (2)): the observational error of the less attentive group biases the parameter(s) of the highly attentive group's forecasting rule.

We summarize these results in the following propositions:

PROPOSITION 1. We propose the cobweb model in which the less attentive group learns from the highly attentive group, but with error. The RPE for the less attentive group varies, based on the size of the variance of the observational errors $\left(\sigma^2_{\tilde{e}_L}\right)$. There are three cases: (i) the exact information observed without observational errors $\left(\sigma^2_{\tilde{e}_L} = 0\right)$, $\bar{\varphi}'_L = (0, 0, 1)$; (ii) the information observed with highly volatile errors ($\sigma^2_{\tilde{e}_L} \rightarrow \infty$), $\bar{\varphi}'_L = \left(\frac{\alpha}{1-\beta}, \frac{\gamma_1}{1-\beta}, 0\right)$; (iii) the information observed with finite error variance ($\sigma^2_{\tilde{e}_L} \in (0, \infty)$), $\bar{\varphi}'_L = \left(\bar{a}_L, \bar{b}_L, \bar{c}_L\right)$, where $\bar{a}_L \in \left(0, \frac{\alpha}{1-\beta}\right)$, $\bar{b}_L \in \left(0, \frac{\gamma_1}{1-\beta}\right)$, and $\bar{c}_L \in (0, 1)$).

PROPOSITION 2 (The Boomerang Effect on the RPE). Consider the cobweb model where the less attentive group learns from the highly attentive group with observational error. The variance of Group L's observational errors $\left(\sigma^2_{\tilde{e}_L}\right)$ affect the equilibrium coefficient of advanced information

$\left(\bar{b}_{2H}\right)$ in Group H's forecast rule ranging between $\left(\frac{\gamma_2}{1-\beta}, \frac{2\gamma_2}{2-\beta}\right)$, for all $\sigma^2_{\tilde{e}_L}$.

III.C. Expectational Stability

To obtain the E-stability condition, we derive the Jacobian matrix from equation (11). The system is E-stable at $\bar{\varphi}$ if and only if all eigenvalues of the Jacobian matrix have negative real parts. The eigenvalues λ_i are[8]:

$$\begin{aligned}
\lambda_1 &= \frac{1}{4}\left(-4+2\beta+\beta c_L - \left(\frac{\beta\left(4\left(\beta\sigma^2_{\tilde{e}_L} - 2b_{2H}\gamma_2\sigma^2_{w2}\right)+\beta\left(\sigma^2_{\tilde{e}_L}+b^2_{2H}\sigma^2_{w2}\right)c_L^2\right)}{\sigma^2_{\tilde{e}_L}+b^2_{2H}\sigma^2_{w2}}\right)^{\frac{1}{2}}\right) \\
\lambda_2 &= \frac{1}{4}\left(-4+2\beta+\beta c_L + \left(\frac{\beta\left(4\left(\beta\sigma^2_{\tilde{e}_L} - 2b_{2H}\gamma_2\sigma^2_{w2}\right)+\beta\left(\sigma^2_{\tilde{e}_L}+b^2_{2H}\sigma^2_{w2}\right)c_L^2\right)}{\sigma^2_{\tilde{e}_L}+b^2_{2H}\sigma^2_{w2}}\right)^{\frac{1}{2}}\right) \\
\lambda_3 &= \frac{1}{4}\left(-4+2\beta+\beta c_L - \frac{\left(\beta\left(8b_{2H}\sigma^2_{w2}\left(\gamma_2\sigma^2_{\tilde{e}_L}+\beta b_{2H}\sigma^2_{\tilde{e}_L}-b^2_{2H}\gamma_2\sigma^2_{w2}\right)+\beta\left(\sigma^2_{\tilde{e}_L}+b^2_{2H}\sigma^2_{w2}\right)c_L^2\right)\right)^{\frac{1}{2}}}{\sigma^2_{\tilde{e}_L}+b^2_{2H}\sigma^2_{w2}}\right) \\
\lambda_4 &= \frac{1}{4}\left(-4+2\beta+\beta c_L + \frac{\left(\beta\left(8b_{2H}\sigma^2_{w2}\left(\gamma_2\sigma^2_{\tilde{e}_L}+\beta b_{2H}\sigma^2_{\tilde{e}_L}-b^2_{2H}\gamma_2\sigma^2_{w2}\right)+\beta\left(\sigma^2_{\tilde{e}_L}+b^2_{2H}\sigma^2_{w2}\right)c_L^2\right)\right)^{\frac{1}{2}}}{\sigma^2_{\tilde{e}_L}+b^2_{2H}\sigma^2_{w2}}\right).
\end{aligned} \tag{14}$$

We conjecture that the E-stability condition is $\beta < 1$ for all $\sigma^2_{\tilde{e}_L}$ in this model.[9] This stability condition gives an equivalent result to the one in which both groups are learning independently. As long as this condition holds, both groups can learn the RPE. This learning occurs regardless of the accuracy with which the less attentive group obtains the highly attentive group's expectations.

To show $\beta < 1$ is the E-stability condition, we separate the values of β into four intervals, $\beta < 0$, $\beta = 0$, $0 < \beta < 1$, and $\beta > 1$. We then consider the first two cases. If $\beta < 0$, the expressions with the square roots become imaginary numbers. The real parts of all λ's < 0. When $\beta = 0$ expectations have no effect on the actual variable of interest, all eigenvalues (14) are equal to -1. This case also satisfies the E-stability condition.

We combine the last two cases into $\beta > 0$. Figures I and II give the general idea of the eigenvalues that we get by changing β given different values of the underlying parameters.[10] Since the results for the eigenvalues (14) are nonlinear and those variables depend on each other, we see that, given the values of other parameters, all real parts of the eigenvalues are negative only if

$0 < \beta < 1$. Based on these numerical results, we believe that our conjecture holds generally that both Groups H and L's estimate of φ_t locally converges to the RPE $(12) - (13)$ if $\beta < 1$.

[INSERT FIGURES I & II]

III.D. Simulations

To illustrate the results we use simulations with reduced form parameters $\alpha = 5$, $\beta = -0.5$, $\gamma_1 = 2$, $\gamma_2 = 2$ in equation (2). There are three objectives for our simulations. First, we examine how the less attentive group's forecasting errors affect the highly attentive group's predictions. Recall that this is the boomerang effect. Second, we verify that $\bar{c}_L$ can be positive for the less attentive group as $t \to \infty$. In particular we examine the extent to which the less attentive group relies on the expectations of the highly attentive group asymptotically (i.e., $\bar{c}_L > 0$ for all finite $\sigma^2_{\tilde{e}_L}$). Finally, we consider how the equilibrium changes as the variation of the observation errors $\left(\sigma^2_{\tilde{e}_L}\right)$ increase. Depending on how accurately they obtain it, the less attentive group uses the observed information from the highly attentive group.

The observable x_t and w_{2t} are one-dimensional normal white noise processes with standard deviation two. The unobservable white noise processes $\tilde{e}_{L,t-1}$ and η_t have standard derivations two and one, respectively. We simulate equations (10) and (18) for both groups. Since adaptive learning is used in the context of local convergence, we assign the initial values which are close to the RPE. The initial values of the PLM for Group L are $a_{L,0} = 1, b_{L,0} = 2$,and $c_{L,0} = 1$, and $R_{L,0}$ is a 3×3 identity matrix. The initial values of the PLM for Group H are $a_{H,0} = 1, b_{1H,0} = 2$,and $b_{2H,0} = 2$, and $R_{H,0}$ is a 3×3 identity matrix. In this simulation, the virtual time period is 10,000.

1. *Results when $cov\,(x_t, w_{2,t}) = 0$*: For Group H, Figure III (left panel) shows that the coefficients $\bar{a}_H$ and $\bar{b}_{1H}$ converge to the REE values 3.333 and 1.333, respectively. On the other hand, $\bar{b}_{2H}$, which is influenced by Group L, converges to 1.42 after period 2,000. This outcome supports the

boomerang effect hypothesis that we discuss above. Group H does not reach the REE for this particular variable ($\bar{b}_{2H} = 1.33$).

The simulation results for Group L are more detailed. Because the less attentive group mispredicts the highly attentive group's expectations, and because it also has a smaller information set, Group L is willing to use a part of the expectations from Group H (i.e., $\bar{c}_L \to 0.62$). Consequently, Group L puts less weight on their own information so that $\bar{a}_L \to 1.24$ and $\bar{b}_L \to 0.51$ (the respective rational expectation equilibrium values $(\bar{a}_L, \bar{b}_L, \bar{c}_L)$ are $(3.33, 1.33, 0)$).

2. Results when $cov\,(x_t, w_{2,t}) \neq 0$: We also relax the orthogonality condition between x_t and $w_{2,t}$. Due to the complexity of analytical calculation, we simulate this case. The result is that the boomerang effect now extends to additional parameters $(\bar{b}_{1H})$ in the highly attentive groups' RPE.

To show this outcome we consider the same reduced form parameters combined with $cov\,(x_t, w_{2,t}) = .25$ and $\sigma_x^2 = 1$. Figure IV shows that both groups converge to their respective RPE: $\bar{\varphi}_L = (2.32, 1.15, 0.31)$ and $\bar{\varphi}_H = (3.33, 1.29, 1.50)$.

[INSERT FIGURES III & IV]

IV. THE BOOMERANG EFFECT

IV.A. Boomerang Effect on the RPE

The simulations in Figures V and VI confirm that both groups can learn the restrictive perceptions equilibrium under least squares learning, but the equilibria change as $\sigma_{\tilde{e}_L}$ varies.[11] Figure V shows the learning paths of both groups for small observational errors ($\sigma_{\tilde{e}_L} = 0.1$) given $\beta = 0.7$. By solving equations (12) and (13), the restrictive perception equilibrium is $\bar{\varphi}_H = (16.67, 6.67, 6.67)$ and $\bar{\varphi}_L = (0, 0, 1)$ for $\sigma_{\tilde{e}_L} = 0.1$.

On the other hand, if the observational error is relatively large ($\sigma_{\tilde{e}_L} = 20$), the restrictive perceptions equilibria for both groups are $\bar{\varphi}_H = (16.67, 6.67, 3.34)$ and $\bar{\varphi}_L = (14.22, 5.69, 0.15)$. Group L reduces its reliance ("weight") on the observed information from Group H ($\bar{b}_L \rightarrow 0.15$). We note that the size of $\sigma_{\tilde{e}_L}$ does not influence the parameters $\bar{a}_H$ and $\bar{b}_{1H}$ for the highly attentive group. However, Figure VI shows that $\bar{b}_{2H}$ changes from 6.67 to 3.34. For the less attentive group, $\bar{a}_L$, $\bar{b}_L$, and $\bar{c}_L$ all change markedly. Clearly, the equilibria are sensitive to the accuracy of the less attentive group's forecast of Group H's expectations.

[INSERT FIGURES V & VI]

Since the size of $\sigma_{\tilde{e}_L}$ influences the differences in $\bar{\varphi}_H$ and $\bar{\varphi}_L$, we now vary its size in a more systematic way. This finding represents the varying ability that Group L has in predicting the highly attentive group's expectations. If Group L is able to obtain Group H's expectations correctly and consistently, $\sigma_{\tilde{e}_L}$ will be close to zero. Otherwise, if the less attentive group consistently makes mistakes in obtaining Group H's expectations, $\sigma_{\tilde{e}_L}$ will be large.

We use the same reduced form parameter values as in Figure III. Again, we assume $cov\left(x_t, w_{2,t}\right) = 0$. The process of the simulation is as follows: There are 200 simulations to which we assign different values of $\sigma_{\tilde{e}_L}$. In $\sigma_{\tilde{e}_L}$, the standard deviation is between 0.1 and 20. Each simulation has a time period of 20,000. When $t = 20,000$, we report $\varphi_{H,t}$ and $\varphi_{L,t}$ as the equilibrium in each simulation.[12]

Figure VII represents the change in coefficients of both groups' PLMs that we obtain by changing the values of $\sigma_{\tilde{e}_L}$. In Figure VII, the x-axis represents the values of the standard deviation of Group L's forecast of Group H's expectations (high expectation in the Figures). The values range from one to 200, where one represents $\sigma_{\tilde{e}_L} = 0.1$, and so on. The first two graphs of the left panel show that the values of $\bar{a}_H$ and $\bar{b}_{1H}$ are around their rational expectations equilibria: $\bar{a}_H = 3.333$ and $\bar{b}_{1H} = 1.333$. The mistakes made by Group L do not affect Group H's estimation of the constant coefficient and the coefficient of common information, x_t.

[INSERT FIGURE VII]

However, the last graph of the left panel in Figure VII shows that the size of $\sigma_{\tilde{e}_L}$ affects on $\bar{b}_{2H}$. Starting from the value of 1.333 for $\sigma_{\tilde{e}_L} = 0.1$, $\bar{b}_{2H}$ trends upward and approaches 1.6 asymptotically as $\sigma_{\tilde{e}_L}$ increases. The right-hand panel shows the change of $\bar{\varphi}_L$ that corresponds to the values of $\sigma_{\tilde{e}_L}$. From top to bottom, the right-hand panel represents the coefficients of $\bar{a}_L$, $\bar{b}_L$ and $\bar{c}_L$, respectively. Not surprisingly, $\bar{a}_L$ and $\bar{b}_L$ run in the same direction from zero to 3.333 and from zero to 1.333, respectively, and $\bar{c}_L$ moves in the opposite direction from one to zero as values of $\sigma_{\tilde{e}_L}$ increase.

IV.B. The Boomerang Effect on the MSE

It is clear that the less attentive group places some weight on the observed information from the highly attentive group. Group L makes use of Groups H's expectations so long as Group L does not have a large variation in observation error in obtaining Group H's information. We now consider how both groups' forecast accuracy is affected if the less attentive group uses the observed expectations from the highly attentive group. To show this, we calculate the mean squared error (MSE) for the forecast of Groups L and H respectively:[13]

$$(15)\qquad MSE_L = \begin{cases} \sigma_\eta^2 & \text{if} \quad \sigma_{\tilde{e}_L}^2 = 0 \\ \left[\frac{2\gamma_2(1-c_L)}{2-\beta(1+c_L)}\right]^2 \sigma_{w_2}^2 + \left[\frac{(\beta-2)c_L}{2}\right]^2 \sigma_{\tilde{e}_L}^2 + \sigma_\eta^2 & \text{if} \quad \sigma_{\tilde{e}_L}^2 \in (0,\infty) \\ \left(\frac{2\gamma_2}{2-\beta}\right)^2 \sigma_{w_2}^2 + \sigma_\eta^2 & \text{if} \quad \sigma_{\tilde{e}_L}^2 \to \infty \end{cases}$$

$$(16)\qquad MSE_H = \begin{cases} \sigma_\eta^2 & \text{if} \quad \sigma_{\tilde{e}_L}^2 = 0 \text{ or } \sigma_{\tilde{e}_L}^2 \to \infty \\ \left(\frac{\beta c_L}{2}\right)^2 \sigma_{\tilde{e}_L}^2 + \sigma_\eta^2 & \text{if} \quad \sigma_{\tilde{e}_L}^2 \in (0,\infty) \end{cases},$$

where $MSE_i \equiv MSE\left(y_{i,t+1|t}^*\right)$ for $i \in \{L, H\}$.

Equation (15), which uses different values of $\sigma_{\tilde{e}_L}^2$, depicts the accuracy of the less attentive group's predictions. According to the first expression of equation (15), where $\sigma_{\tilde{e}_L}^2 = 0$, Group L is able to observe the expectations from Group H without any observation errors. The result is that

Group L obtains the minimum MSE ($MSE_L = \sigma_\eta^2$). However, omitted variable and observation error problems reduce the Group L's predictive accuracy. This outcome can be seen in the second and third conditions of equation (15). The variances of the omitted variable $\left(\sigma_{w_2}^2\right)$ and observation errors $\left(\sigma_{\tilde{e}_L}^2\right)$ appear in those conditions and make the MSE larger than σ_η^2.

The results for Group H indicate that only the two limit points of the variance of the observation errors $\left(\sigma_{\tilde{e}_L}^2 = 0 \text{ or } \sigma_{\tilde{e}_L}^2 \to \infty\right)$ produce the most efficient outcome. However, if Group L observes and uses the expectations from the Group H (with finite observation error variance) to form its estimation, then Group H's MSE is also affected, becoming less efficient. Again, this consequence is *the boomerang effect on the MSE*:

PROPOSITION 3 (The Boomerang Effect on the MSE). Consider the cobweb model where the less attentive group learns from the highly attentive group with observational error. The finite variance of Group L's observational errors $\left(\sigma_{\tilde{e}_L}^2\right)$ generates higher mean square error (MSE) in Group H in a range of $\frac{\gamma_2}{1-\beta}$ and $\frac{2\gamma_2}{2-\beta}$.

Figure VIII displays the relation between both groups' MSEs and the variance of observation errors. The values of parameters are $\beta = -0.5,\ \gamma_2 = 2, \sigma_{w_2}^2 = 4, \sigma_\eta^2 = 1,$ and $\sigma_{\tilde{e}_L}^2 \in [0, 400]$. In the upper panel of Figure VIII, we see that the MSE for the less attentive group is higher as $\sigma_{\tilde{e}_L}^2$ increases. It converges to 11.24 as $\sigma_{\tilde{e}_L}^2 \to \infty$. For the lower panel, Figure VIII graphs equation (16). When $\sigma_{\tilde{e}_L}^2 = 0$, the MSE for the highly attentive group is unity $\left(\sigma_\eta^2 = 1\right)$. If $\sigma_{\tilde{e}_L}^2$ increases, MSE_H increases up a point then decreases asymptotically to 1 as $\sigma_{\tilde{e}_L}^2 \to \infty$.

[INSERT FIGURE VIII]

V. Discussion

We have noted a boomerang effect when the less attentive group faces this omitted variables problem. The transparent, expected result is that Group L makes less accurate predictions if they lack full information. Because the model is based on social interactions and is self-referential, the less attentive group's incorrect expectations affect the highly attentive group's expectations and forecast accuracy. The boomerang effect on Group H's predictions occurs since the highly attentive group cannot disentangle the forecast errors of Group L contemporaneously. This situation occurs despite the fact that Group H has the full information set.

In our model (13), the less attentive group's failure to obtain the information contained in $(w_{2,t})$ directly affects the range of values that coefficient $\bar{b}_{2H}$ takes. But, since both groups have a common information set $(1, x_{t-1})$, Group L can actually use the information at hand to make its predictions. Thus, the boomerang effect does not occur on the constant coefficient and the coefficient of x_t, but it can extend to other coefficients (parameters) when $cov\,(x_t, w_{2,t}) \neq 0$.

The boomerang effect not only exists on Group H's coefficient(s), but also affects their predictive accuracy. Equation (16) shows that the mean squared error for Group H becomes larger. We note two extreme cases, $\sigma^2_{\tilde{e}_L} = 0$ and $\sigma^2_{\tilde{e}_L} \to \infty$. When $\sigma^2_{\tilde{e}_L} = 0$, the less attentive group, in the limit, uses the expectations from the highly attentive group. We know that Group L has no observation error in this case; their expectations become exactly the same as those of Group H. Both groups are the same group in the limit, which implies that they have all the information they need to form their expectations. Both groups' MSEs are at the minimum $\left(MSE_L = MSE_H = \sigma^2_\eta\right)$.

However, if $\sigma^2_{\tilde{e}_L} \to \infty$, the less attentive group eventually realizes that the highly attentive group's expectations are observed with noise and discards them. Both groups learn independently. Group L has a smaller information set that reduces it predictive accuracy (15). But there is no boomerang effect, since both groups learn independently and Group H has all available information $\left(MSE_H = \sigma_\eta^2\right)$.

In sum, for the case of finite observation errors $\left(\sigma_{\tilde{e}_L}^2 \in (0,\infty)\right)$, the predictive accuracy of Group H's forecasts are further impaired. We know that equation (2) is a self-referential model in which the expectations of y_t — $(E_{t-1}^{*} y_t)$ — of all agents affect the actual y at time t. The presence of observation error leads the less attentive group to form an incorrect expectation via the variable $\hat{y}_{t-1}$ in equation (7). This relation alters the actual y_t in equation (10), so when the highly attentive group forms its new expectations of y_{t+1} at time t, using $\varphi_{H,t}$, there are errors. Group H fails to forecast the actual y_{t+1} correctly. Again, their mean squared error is larger than σ_η^2 when $\sigma_{\tilde{e}_L}^2 \in (0,\infty)$.

VI. Conclusion and Implications

In this paper, we explore the case of social interaction where a less attentive and less informed group learns and interacts with a more attentive and more informed group. We focus particular attention on the boomerang effect, which we define as a situation in which the inaccurate forecasts of a less attentive group confound a more attentive group's forecasts. To determine the boomerang effect, we construct models that allow for the diffusion of information between the less attentive (Group L) and highly attentive (Group H) agents.

We assume that the less attentive agents have only part of the available information, but the highly attentive agents have more, or even full, information that they can use to predict the variable of interest. Group L also observes the expectations (with errors) of Group H. In this situation, it is advantageous for Group L to use the observed expectations from Group H, since the less attentive group does not have all available information to make correct predictions.

However, although using the expectations of the highly attentive group is necessary, the results depend on whether Group L is able to obtain accurately these expectations. If the observed expectations from Group H are accurate, then the less attentive group place less weight (asymptotically) on their own information. Instead, Group L relies on the highly attentive group's expectations. If the observed expectations are inaccurate, Group L treats the information as noise and puts no weight on it (asymptotically). In place of Group H's expectations, Group L emphasizes its own information.

The accuracy of this information is also important in that it dictates the magnitude of the boomerang effect. The equilibria are sensitive to how well the less attentive group's forecasts the expectations of the highly attentive group. Not only are the equilibria variable, but also the greater the inaccuracy of Group L's forecasts, the more severe the boomerang effect that confounds the accuracy of the highly attentive group's forecasts.

To the degree that a class of models such as the cobweb model have endogenous variable(s) that depend on agent forecasts (i.e., are self referential), and where information heterogeneity is present, these findings are pertinent. Among the potential implications, two are noted. The first implication centers on the role the boomerang effect has in macroeconomic models that traditionally rely on institutional frictions such as contracts and menu costs. The boomerang effect shifts the focus back to consideration of matters that pertain to models that emphasize informational ambiguities [Lucas, 1972, 1973] and suggests a formal framework on the effects of policy pronouncements or policy actions that are aimed at shaping private sector expectations.

The second implication is for policy. The idea of the boomerang effect is important because it points to issues of transparency for policy actions or policy pronouncements that influencw agent expectations and behavior. The boomerang effect suggests that the less attentive public's inaccurate forecasts may adversely affect the forecasts of the more attentive and better informed policymakers. Greater transparency by policymakers in the procedures they follow and in the information they provide would be the appropriate response in maximizing forecast accuracy for all.

Appendix 1: Stability of Adaptive Learning

Expectational Stability

This appendix briefly introduces the stability properties of adaptive learning process (expectational stability or E-stability). We base our tests for learning on the almost surely convergence of the PLM and ALM parameters to the REE. Using the ALM's and PLM's one can devise a condition for mapping the PLM to the ALM. The issue concerning learning is whether, in the limit, there is convergence in some suitable stochastic sense of the forecast equation parameters to the REE. The E-stability condition determines if agents do learn the correct forecasting rule, which is the REE.

Based on the results of Marcet and Sargent [1989a,b], Evans [1989], and Evans and Honkapohja [1992], the E-stability condition is defined in terms of the ordinary differential equation (ODE):

$$\frac{d\theta}{d\tau} = T(\theta) - \theta, \tag{17}$$

where θ is a finite dimensional parameter specified in the perceived laws of motion, $T(\theta)$ is a mapping (so-called T-mapping) from the perceived to the actual laws of motion, and τ denotes notional or artificial time. The REE $\bar{\theta}$ corresponds to fixed points of $T(\theta)$.

Expectational stability and adaptive learning (or least squares learning) rules provide tightly related approaches to solve the stability problems for rational expectations equilibria. However, Evans and Honkapohja [1999, 2001] argue that the convergence of the econometric learning involves more technical analysis.

Least Squares Learning and Recursive Stochastic Algorithms

We assume that agents use recursive least squares (RLS) for updating their expectations.

They update each period up to the period $t-1$ [Bray, 1982; Bray and Savin, 1986; Marcet and Sargent, 1989a,b] Both groups use a least squares regression of y_t on $z_{L,t-1}$ and $z_{H,t-1}$ to estimate $\varphi_{L,t-1}$ and $\varphi_{H,t-1}$, respectively, given the data through time $t-1$.

Applying the RLS formula, we obtain the updating mechanism:

$$\begin{aligned}\varphi_{i,t} &= \varphi_{i,t-1}+(t+T_i)^{-1}R_{i,t}^{-1}z_{i,t-1}(y_t-\varphi'_{i,t-1}z_{i,t-1}) \\ R_{i,t} &= R_{i,t-1}+(t+T_i)^{-1}(z_{i,t-1}z'_{i,t-1}-R_{i,t-1}),\end{aligned} \tag{18}$$

where $i\in\{L,H\}$. This multivariate version of the recursive algorithm reduces to least squares with specified initial conditions for some appropriate values of T_i, $\varphi_{i,0}$, and $R_{i,0}$.

Determining asymptotic stability for a general class of multivariate linear models (convergence of parameters to some point) requires the convergence conditions of stochastic recursive algorithm(s) (SRA) given by Ljung [1977]. We can determine the convergence by forming the system (10) and (18) as the standard SRA [Marcet and Sargent, 1989a,b; Evans and Honkapohja, 1996]:

$$\theta_t=\theta_{t-1}+\delta_{i,t}Q(\theta_{t-1},X_{i,t},t), \tag{19}$$

where $\theta'_t=(vec(\varphi_{L,t})',vec(\varphi_{H,t})',vec(R_{L,t+1}),vec(R_{H,t+1})),X_{i,t}=(z_{i,t},z_{i,t-1},\eta_t)$, and $\delta_t=(t+T_i)^{-1}$.

The limit points of the SRA relate to the local equilibria of the ODE [Evans and Honkapohja, 2001, p. 35]:

$$\frac{d\theta}{d\tau}=h\left(\theta\left(\tau\right)\right),$$

where $h\left(\theta\right)$ is obtained as:

$$h\left(\theta\right)=\lim_{t\to\infty}EQ\left(\theta,\bar{X}_t\left(\theta\right),t\right),$$

provided this limit exists. E denotes the expectations of $Q\left(\theta,X_t,t\right)$ for fixed θ. $X_t\left(\theta\right)$ is the stochastic process for X_t, which we can obtain by holding θ_{t-1} at the fixed value $\theta_{t-1}=\theta$. Following the set-up of the SRA, $\bar{\theta}$ is an equilibrium point of $\frac{d\theta}{d\tau} = h\left(\bar{\theta}\right)$ if $h\left(\bar{\theta}\right) = 0$. The local stability of $\bar{\theta}$ follows when all eigenvalues of $Dh\left(\bar{\theta}\right)$ have negative real parts [Evans and Honkapohja, 2001, p. 34–43].

Appendix 2: The Case of Homogeneous Information Sets

Our working hypothesis is that the boomerang effect only occurs for heterogeneous information levels. The case of homogeneous information provides a baseline for comparison. For the homogeneous information set case, we assume that two groups are learning the true model, and that Group L uses its own information and the expectations (with errors) from Group H. We also assume that both groups have the same information set from which they form their learning processes.

Not surprisingly, the learning dynamics are such that the less attentive agents eventually discard the information from the highly attentive group, since both groups have the same information set. Moreover, the stability condition in this model gives an equivalent result to the case of heterogeneous information sets. We use simulations to confirm the learning process in which Group L depends on the expectations of Group H for some period of time. However, this information is discarded asymptotically.

We assume that Group L learns from Group H so that their PLM is:

$$y_t = a_{L,t-1} + b'_{L,t-1} w_{t-1} + c_{L,t-1} \widehat{y}_{t-1} + v_t \tag{20}$$

The PLM for the highly attentive group, Group H, excludes any information from the less attentive group,

$$y_t = a_{H,t-1} + b'_{H,t-1} w_{t-1} + v_t. \tag{21}$$

Now suppose both groups forecast following the process of equations (20) and (21), respectively, and they have data on the political economic system from periods $t_i = T_i, ..., t-1$, where $i \in \{L, H\}$. Therefore, the time $t-1$ information set for the less attentive group, Group L, is $\{y_i, w_i, \widehat{y}_i\}_{i=T_L}^{t-1}$ but the information set for the highly attentive group, Group H, at time $t-1$ is

$\{y_i, w_i\}_{i=T_H}^{t-1}$. The two groups use (22) and (23), respectively, to forecast the variable of interest:[14]

$$E^*_{L,t-1} y_t = \varphi'_{L,t-1} z_{L,t-1} \tag{22}$$

and

$$E^*_{H,t-1} y_t = \varphi'_{H,t-1} z_{H,t-1}, \tag{23}$$

where $\varphi'_{L,t-1} = \begin{pmatrix} a_{L,t-1} & b'_{L,t-1} & c_{L,t-1} \end{pmatrix}$, $\varphi'_{H,t-1} = \begin{pmatrix} a_{H,t-1} & b'_{H,t-1} \end{pmatrix}$, $z_{L,t-1} = \begin{pmatrix} 1 \\ w_{t-1} \\ \widehat{y}_{t-1} \end{pmatrix}$, and $z_{H,t-1} = \begin{pmatrix} 1 \\ w_{t-1} \end{pmatrix}$.

Since we assume that both groups are evenly distributed, we can use the true model (2). We can obtain the actual law of motion (ALM) by substituting (22) and (23) into (2), yielding:

$$y_t = \Phi' z_{t-1} + \eta_t, \tag{24}$$

where $\Phi' = \begin{pmatrix} \phi_1 & \phi'_2 & \phi_3 \end{pmatrix} = \begin{pmatrix} \alpha + \beta\left(\frac{a_{L,t-1}+a_{H,t-1}}{2}\right) & \beta\left(\frac{b'_{L,t-1}+b'_{H,t-1}}{2}\right) + \gamma' & \beta\left(\frac{c_{L,t-1}}{2}\right) \end{pmatrix}$ and $z_{t-1} = \begin{pmatrix} 1 \\ w_{t-1} \\ \widehat{y}_{t-1} \end{pmatrix}$.

We use equations (2), (18), (20) and (21) to compute the associated differential equation for the stochastic recursive algorithm:

$$\frac{d\varphi}{d\tau} = \begin{pmatrix} \alpha \\ \gamma \\ 0 \\ \alpha \\ \gamma \end{pmatrix} + \frac{1}{2}\beta \begin{pmatrix} a_L + a_H \\ b_L + b_H \\ c_L \\ a_L + a_H + c_L a_H \\ b_L + b_H + b_H c_L \end{pmatrix} - \begin{pmatrix} a_L \\ b_L \\ c_L \\ a_H \\ b_H \end{pmatrix}, \tag{25}$$

where $\varphi = \begin{pmatrix} \varphi_L \\ \varphi_H \end{pmatrix}$.

We calculate the unique REE by setting (25) equal to zero. The resulting identities for the less and highly attentive groups (respectively) are:

$$\begin{aligned} \bar{a}_L &= \frac{\alpha}{1-\beta} \\ \bar{b}_L &= \frac{\gamma}{1-\beta} \\ \bar{c}_L &= 0 \end{aligned}$$

and

$$\bar{a}_H = \frac{\alpha}{1-\beta}$$
$$\bar{b}_H = \frac{\gamma}{1-\beta}.$$

We obtain the E-stability condition using the associated ODE of (25) and solve its Jacobian matrix. We then calculate the eigenvalues of the Jacobian matrix, given the REE. Those eigenvalues are $\lambda_1 = \frac{\beta}{2} - 1$, $\lambda_2 = \frac{1}{4}\left(-4 + \beta(2 + c_L) - \beta\sqrt{4 + c_L^2}\right)$, and $\lambda_3 = \frac{1}{4}\left(-4 + \beta(2 + c_L) + \beta\sqrt{4 + c_L^2}\right)$. Substituting the REE, $c_L = 0$, we can then express the eigenvalues in (26):

$$\begin{aligned} \lambda_1 &= \tfrac{\beta}{2} - 1 \\ \lambda_2 &= -1 \\ \lambda_3 &= -1 + \beta. \end{aligned} \tag{26}$$

According to the previous section, we know that $\bar{\varphi}$ is locally stable if all eigenvalues $(\lambda' s)$ have negative real parts. From the matrix (26), it follows that the system is E-stable if $\beta < 1$.

PROPOSITION A. We propose a cobweb model, in which the less attentive group, Group L, and the highly attentive group, Group H, have the same (full) information set and Group L uses the observed information $(\widehat{y}_{t-1})$ in forecasting the variable of interest. For both groups, provided that $\beta < 1$, φ_t converges locally to the REE $\bar{\varphi}$.

The REE shows that Group L overparameterizes its learning mechanism by using the expectations from Group H. If both groups learn independently, the E-stability condition is $\beta < 1$. Here, the overparameterization by the less attentive group does not change the E-stability condition. In this sense the REE is strongly E-stable and thus corresponds to the adaptive learning rule.[15]

We consider that both groups have the same information set w_{t-1} and that the less attentive group uses the variable $\widehat{y}_{t-1}$ to assist in their forecasts. But, since $\widehat{y}_{t-1}$ contains observational error, Group L agents put less and less weight on $\widehat{y}_{t-1}$ as they increasingly use the information set

w_{t-1} ($c_L = 0$ in the REE). Both groups learn the same REE, even though Group L learns initially from Group H.

From equation (11), we derive the Jacobian matrix:

$$\begin{pmatrix} \frac{(\beta-2)}{2} & 0 & 0 & \frac{\sigma_{\tilde{e}_L}^2\beta-2\gamma_2\sigma_{w2}^2}{2\left(\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2\right)} & 0 & \frac{a_H\sigma_{w2}^2\left(-\gamma_2\sigma_{\tilde{e}_L}^2-\beta b_{2H}\sigma_{\tilde{e}_L}^2+\gamma_2 b_{2H}^2\sigma_{w2}^2\right)}{\left(\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2\right)^2} \\ 0 & \frac{(\beta-2)}{2} & 0 & 0 & \frac{\sigma_{\tilde{e}_L}^2\beta-2\gamma_2\sigma_{w2}^2}{2\left(\sigma_{\tilde{e}_L}^2+\sigma_{w2}^2 b_{2H}^2\right)} & \frac{b'_{1H}\sigma_{w2}^2\left(-\gamma_2\sigma_{\tilde{e}_L}^2-\beta b_{2H}\sigma_{\tilde{e}_L}^2+\gamma_2 b_{2H}^2\sigma_{w2}^2\right)}{\left(\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2\right)^2} \\ 0 & 0 & \frac{(\beta-2)}{2} & 0 & 0 & \frac{\sigma_{w2}^2\left(\gamma_2\sigma_{\tilde{e}_L}^2+\beta b_{2H}\sigma_{\tilde{e}_L}^2+\gamma_2 b_{2H}^2\sigma_{w2}^2\right)}{\left(\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2\right)^2} \\ \frac{\beta}{2} & 0 & \frac{\beta a_H}{2} & \frac{-2+\beta(1+c_L)}{2} & 0 & 0 \\ 0 & \frac{\beta}{2} & \frac{\beta b_{1H}}{2} & 0 & \frac{-2+\beta(1+c_L)}{2} & 0 \\ 0 & 0 & \frac{\beta b_{2H}}{2} & 0 & 0 & \frac{-2+\beta(1+c_L)}{2} \end{pmatrix}.$$

After some computation, the eigenvalues of the matrix are:

$$\lambda_1 = \frac{1}{4}\left(-4+2\beta+\beta c_L-\left(\frac{\beta\left(4\left(\beta\sigma_{\tilde{e}_L}^2-2b_{2H}\gamma_2\sigma_{w2}^2\right)+\beta\left(\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2\right)c_L^2\right)}{\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2}\right)^{\frac{1}{2}}\right)$$

$$\lambda_2 = \frac{1}{4}\left(-4+2\beta+\beta c_L+\left(\frac{\beta\left(4\left(\beta\sigma_{\tilde{e}_L}^2-2b_{2H}\gamma_2\sigma_{w2}^2\right)+\beta\left(\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2\right)c_L^2\right)}{\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2}\right)^{\frac{1}{2}}\right)$$

$$\lambda_3 = \frac{1}{4}\left(-4+2\beta+\beta c_L-\frac{\left(\beta\left(8b_{2H}\sigma_{w2}^2\left(\gamma_2\sigma_{\tilde{e}_L}^2+\beta b_{2H}\sigma_{\tilde{e}_L}^2-b_{2H}^2\gamma_2\sigma_{w2}^2\right)+\beta\left(\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2\right)c_L^2\right)\right)^{\frac{1}{2}}}{\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2}\right)$$

$$\lambda_4 = \frac{1}{4}\left(-4+2\beta+\beta c_L+\frac{\left(\beta\left(8b_{2H}\sigma_{w2}^2\left(\gamma_2\sigma_{\tilde{e}_L}^2+\beta b_{2H}\sigma_{\tilde{e}_L}^2-b_{2H}^2\gamma_2\sigma_{w2}^2\right)+\beta\left(\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2\right)c_L^2\right)\right)^{\frac{1}{2}}}{\sigma_{\tilde{e}_L}^2+b_{2H}^2\sigma_{w2}^2}\right).$$

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ENDNOTES

1. There is substantial empirical support for information heterogeneity [Delli-Carpini and Keeter, 1996; Krause and Granato, 1998; Granato and Krause, 2000; Nie, Junn, and Stehlik-Berry, 1996; MacKuen, Erikson, and Stimson, 1989]. This body of research shows that agents can be separated into two distinct groups. One group, exhibits high information and attention levels (Group H), are "issue publics" or opinion leaders [Lazarsfeld, Berelson, and Gaudet, 1944; Katz and Lazarsfeld, 1955]. Another group, which can be categorized as less attentive (Group L), reacts less swiftly to changes in the political (and economic) environment [Graber, 1984; MacKuen, 1984].

2. Lucas [1973] derives the reduced form of the inflation behavior which is consistent with the cobweb model. An aggregate supply function presents as the following form:

$$y_t = y^n + \varkappa \left(p_t - E^*_{t-1} p_t\right) + \zeta_t, \qquad \varkappa > 0$$

where y_t is the real output level, p_t is the aggregate price level. On the other hand, the aggregate demand function is based on the quantity theory of money:

$$m_t + \nu_t = p_t + y_t,$$

where m_t is the money supply. v_t is velocity which depends on some exogenous variables, w_{t-1}so that:

$$v_t = \varrho + \psi' w_{t-1} + \xi_t.$$

The money supply rule is:

$$m_t = \bar{m} + \phi' w_{t-1} + \varepsilon_t,$$

where ζ_t, ξ_t and ε_t are white noise aggregate shocks. We therefore derive equation (2) from the above equations:

$$y_t = \alpha + \beta E^*_{t-1} y_t + \gamma' w_{t-1} + \eta_t,$$

where $\alpha \equiv (\bar{m} + \varrho - y^n) / (1 + \varkappa)$, $\beta \equiv \varkappa / (1 + \varkappa)$ and $\eta_t \equiv (\xi_t + \varepsilon_t - \zeta_t) / (1 + \varkappa)$.

3. Appendix 1 provides some basic background on adaptive learning. See Evans and Honkapohja [2001] for details.

4. See Evans and Honkapohja [2001] for the case in which the two groups learn the REE independently.

5. See Appendix 2 for the case of the homogenous information set.

6. We note that since the observed information $\widehat{y}_{t-1}$ is predetermined, the expectation operator in equation (7) has no affect on $\widehat{y}_{t-1}$at time $t-1$.

7. If $cov\,(x_t, w_{2,t}) \neq 0$, the simulations (See Figures IV and V) show that $\bar{b}_{1H}$ is also affected by the less attentive group.

8. λ_1 and λ_2 are repeated roots which results in six total eigenvalues. Our 6×6 Jacobian matrix reflects this. See Appendix 3 for the Jacobian matrix and its eigenvalues.

9. We solve that if $\sigma^2_{\tilde{e}_L} \to 0$ or $\sigma^2_{\tilde{e}_L} \to \infty$, it is E-stable for $\beta < 1$. Due to the complexity of this model, we use numerical procedures to show that $\beta < 1$ is also the E-stability condition for $\sigma^2_{\tilde{e}_L} \in (0, \infty)$.

10. The parameters for Figure I are $\left(\gamma_2 = 2, \sigma^2_{w2} = 4, \sigma^2_{\tilde{e}_L} = 4\right)$. The parameters for Figure II are $\left(\gamma_2 = 2, \sigma^2_{w2} = 4, \sigma^2_{\tilde{e}_L} = 100\right)$.

11. The underlying parameters in the simulations are: $\alpha = 5$, $\beta = 0.7$, $\gamma_1 = 2$, $\gamma_2 = 2$, $x_t \sim iid\,(0, 4)$, $w_{2,t} \sim iid\,(0, 4)$, $\tilde{e}_{L,t} \sim iid\,(0, 0.01)$ and $\eta_t \sim iid\,(0, 1)$. The initial values of the

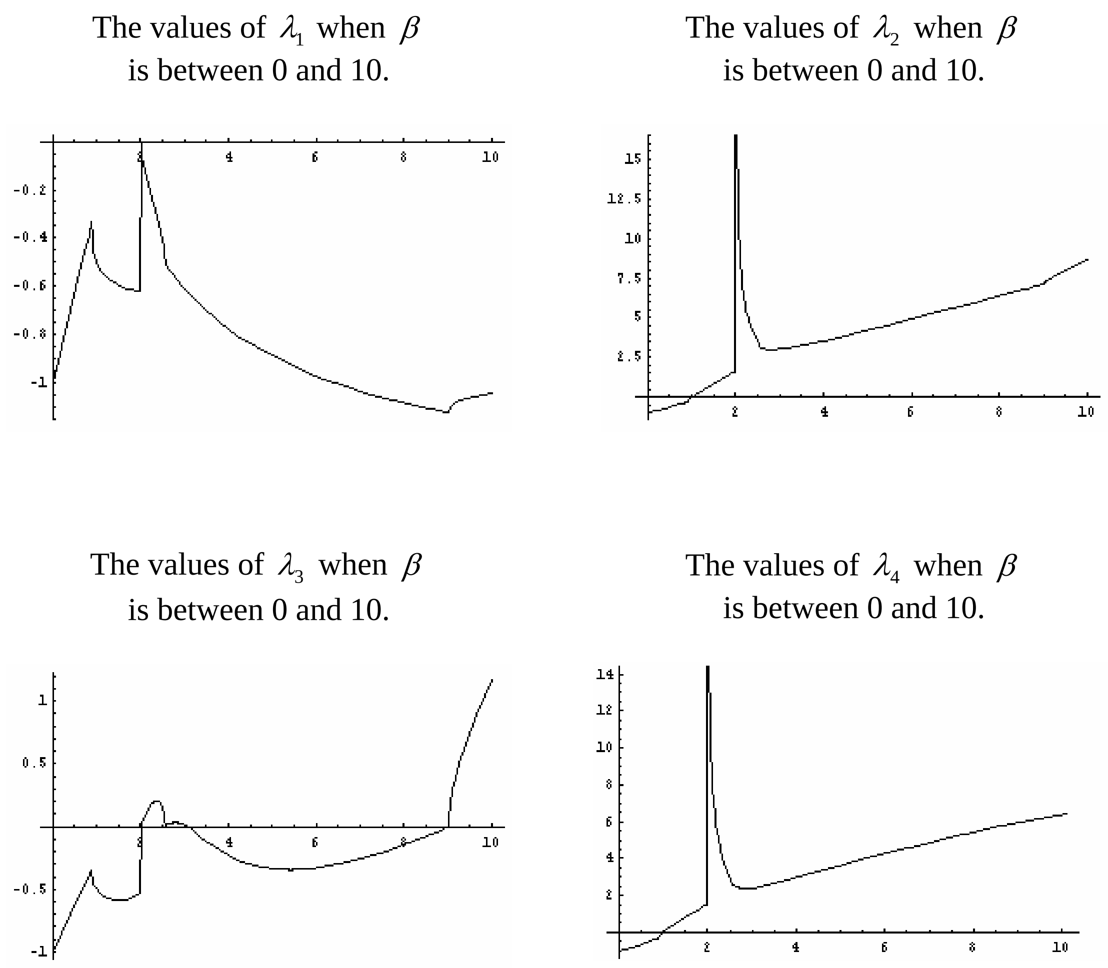
coefficients for both groups are: $\varphi_{L,0} = (0.2, 0.2, 0.9)$ and $\varphi_{H,0} = (16, 7, 7)$. The virtual time period is $20,000$.

12. According to the previous simulations, it is clear that the values converge when $t = 2,000$. To verify that values converge, we report up to $t = 20,000$.

13. For comparison, we also calculate the MSEs for situations in which both groups have the same (full) information set and learn independently. We show that both groups' MSE's are at a minimum when $MSE_L = MSE_H = \sigma_\eta^2$.

14. We note that since the observed information $\widehat{y}_{t-1}$ is predetermined, the expectation operator in equation (22) has no affect on $\widehat{y}_{t-1}$at time $t-1$.

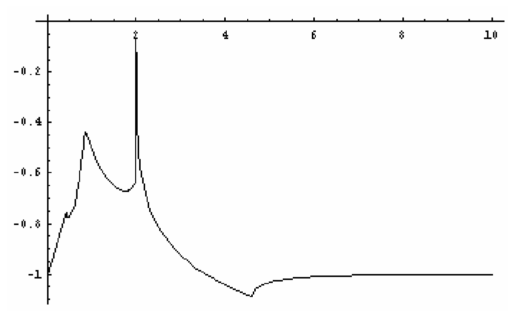
15. We do not follow exactly the standard usage of strong E-stability. Evans and Honkapohja [2001] define an REE as strongly E-stable even when a single group of agents initially overparameterize an REE solution. However, they also suggest that "one can also allow for heterogeneous expectations across agents and determine whether allowing for heterogeneity alters the stability conditions for convergence of adaptive learning." [Evans and Honkapohja, 2001, p. 42].



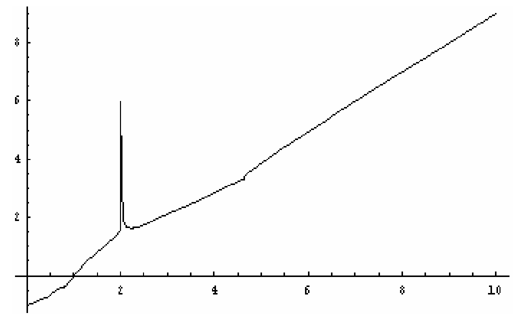
Note: $\gamma_2 = 2$, $\sigma^2_{w_2} = 4$, and $\sigma^2_{\tilde{e}_L} = 4$.

FIGURE 1. EIGENVALUES OF MODEL 2 WITH $\sigma^2_{\tilde{e}_L} = 4$

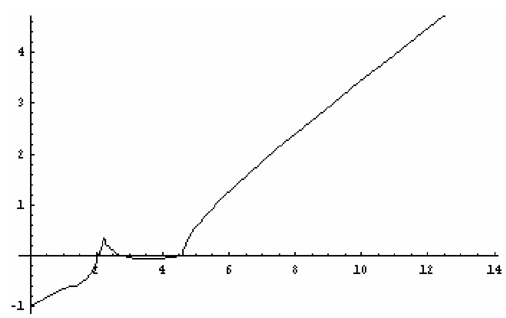
The values of λ_1 when β is between 0 and 10.



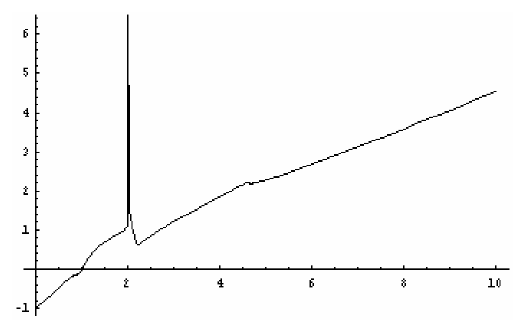
The values of λ_2 when β is between 0 and 10.



The values of λ_3 when β is between 0 and 10.



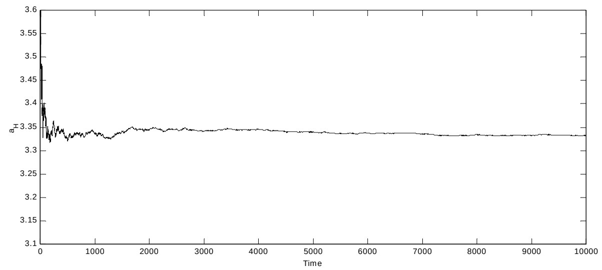
The values of λ_4 when β is between 0 and 10.



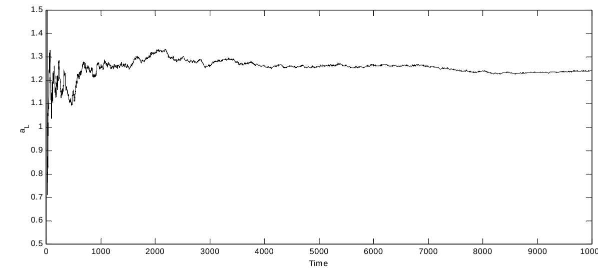
Note: $\gamma_2 = 2$, $\sigma_{w_2}^2 = 4$, and $\sigma_{\tilde{e}_L}^2 = 100$.

FIGURE 2. EIGENVALUES OF MODEL 2 WITH $\sigma_{\tilde{e}_L}^2 = 100$

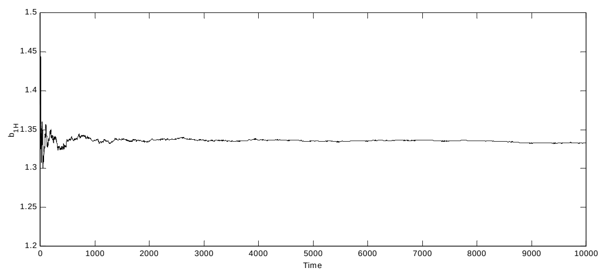
The Constant Coefficient of the Highly Attentive Group's PLM



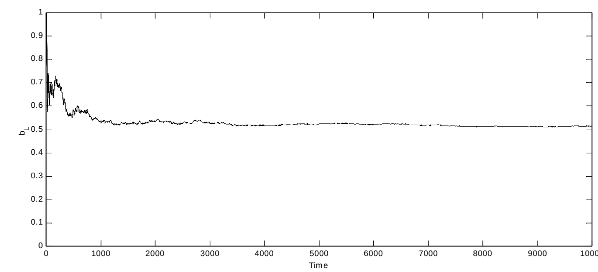
The Constant Coefficient of the Less Attentive Group's PLM



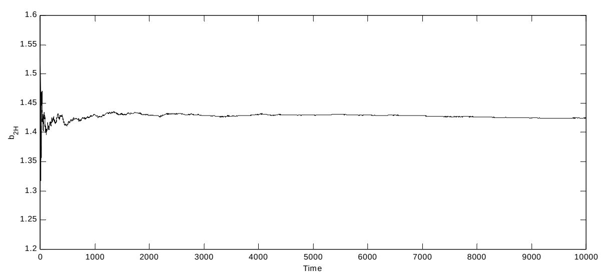
The Coefficient of x_1 for the Highly Attentive Group's PLM



The Coefficient of w_1 for the Less Attentive Group's PLM



The Coefficient of w_2 for the Highly Attentive Group's PLM



The Coefficient of Highly Attentive Group's Expectation for the Less Attentive Group's PLM

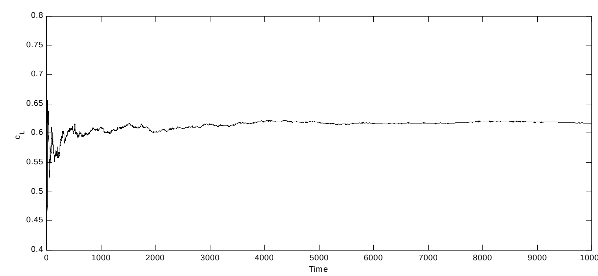


FIGURE 3. SIMULATIONS OF MODEL 2: $cov\ (x_t\ ,w_{2,t})=0$

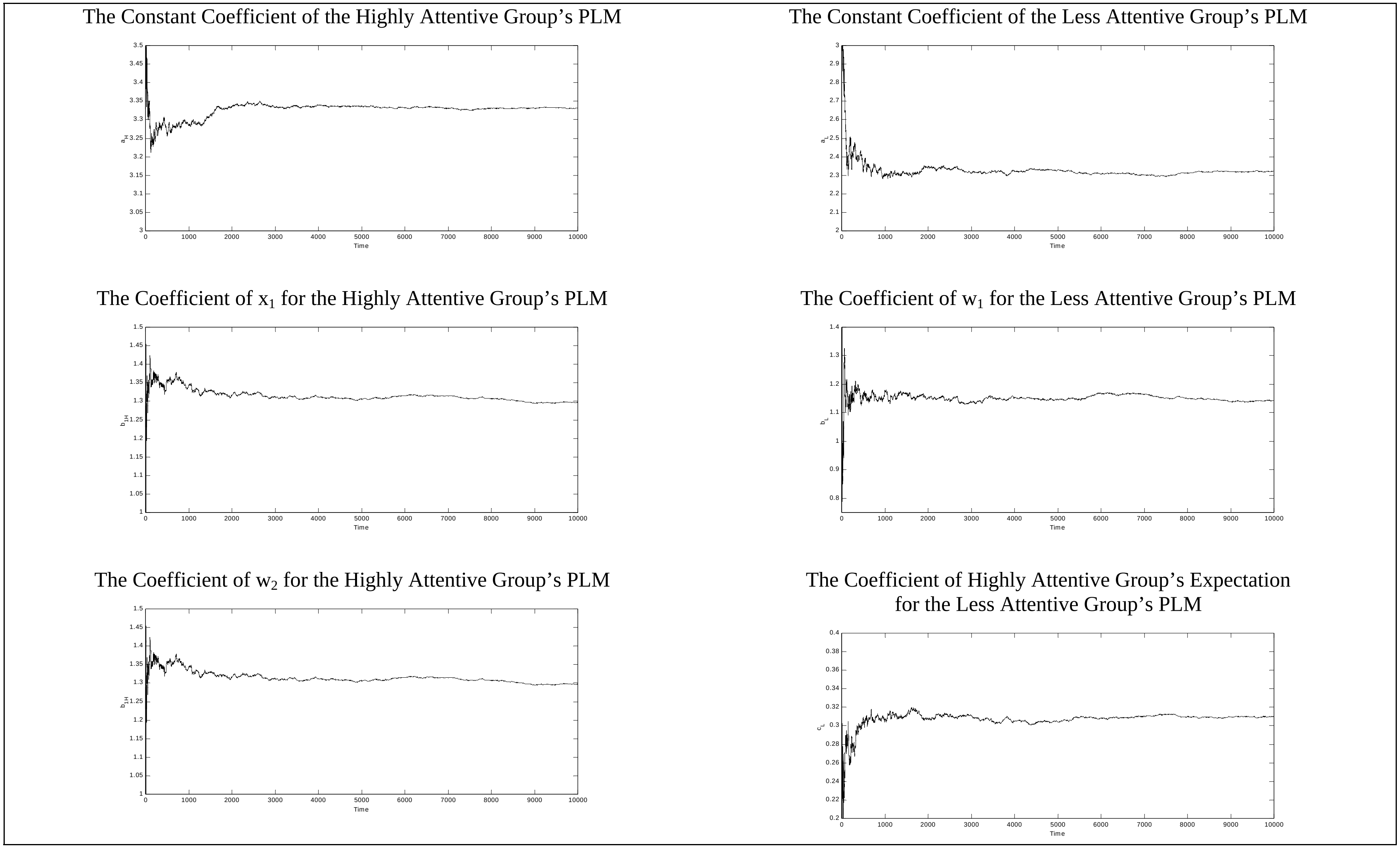


FIGURE 4. SIMULATIONS OF MODEL 2: $cov\ (x_t\ , w_{2,t}) = 0.25$

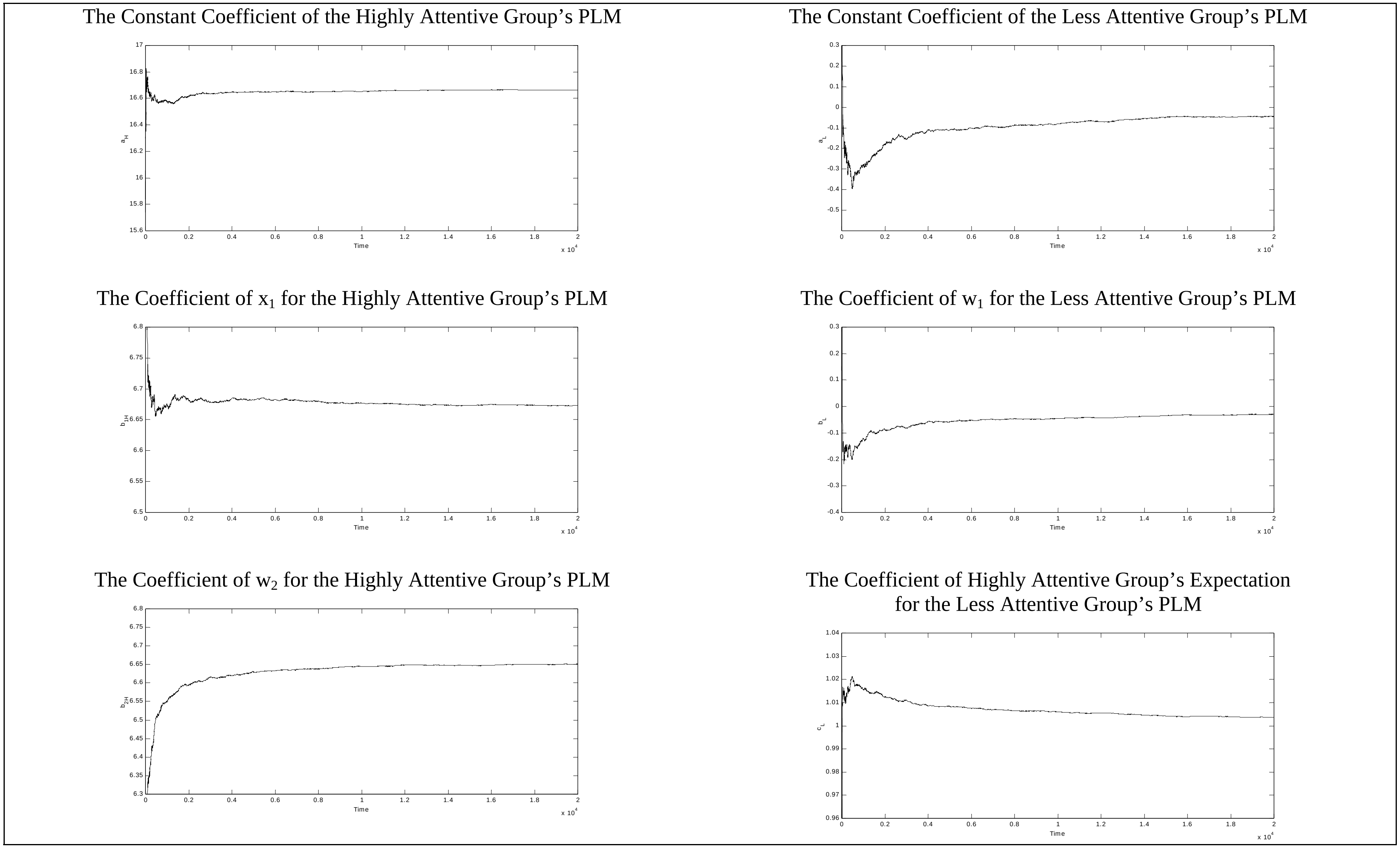


FIGURE 5. SIMULATIONS OF MODEL 2: $\sigma_{\tilde{e}_L} = 0.1$

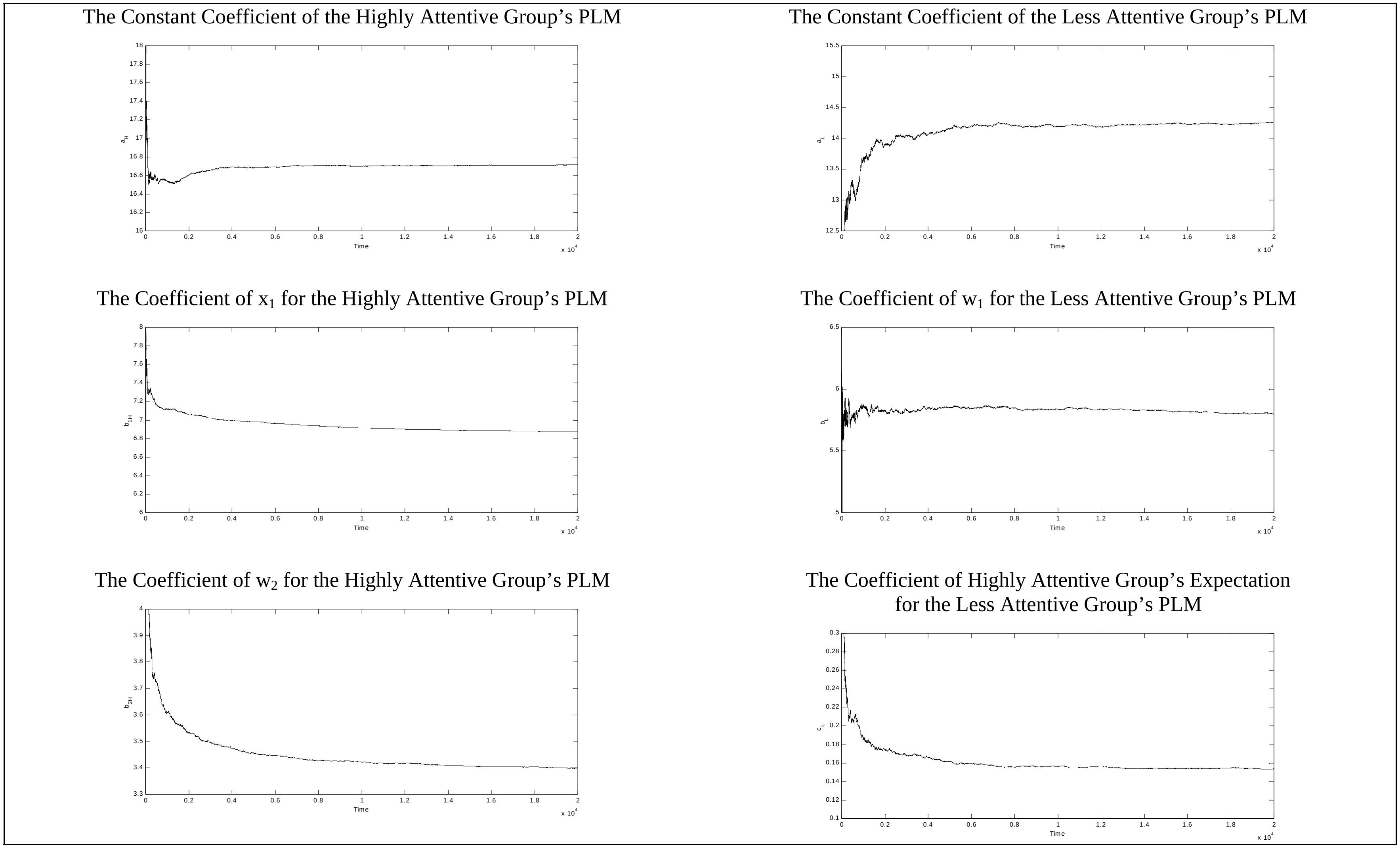


FIGURE 6. SIMULATIONS OF MODEL 2: $\sigma_{\tilde{e}_L} = 20$

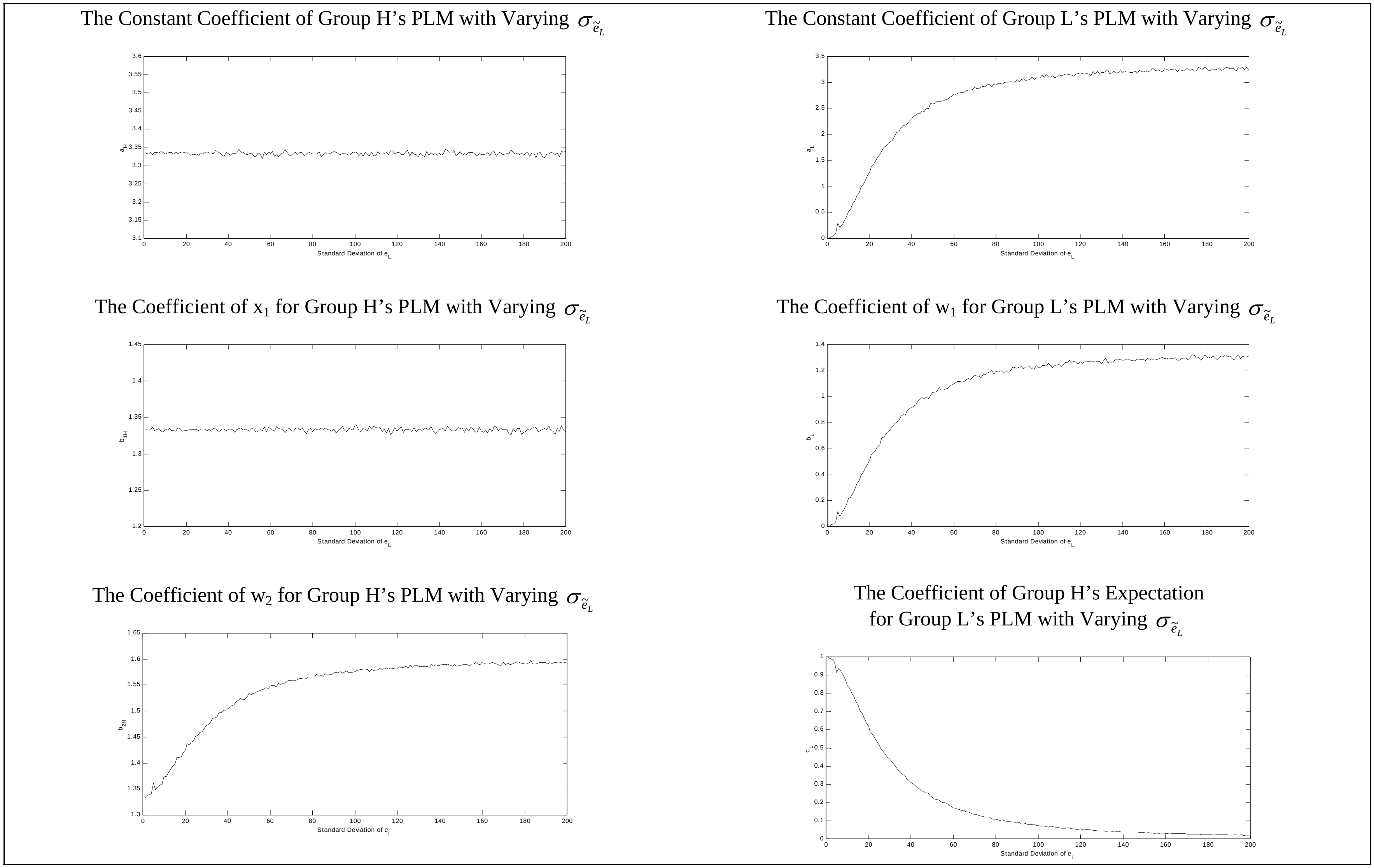


FIGURE 7. SIMULATIONS OF MODEL 2 WITH VARYING $\sigma_{\tilde{e}_L}$

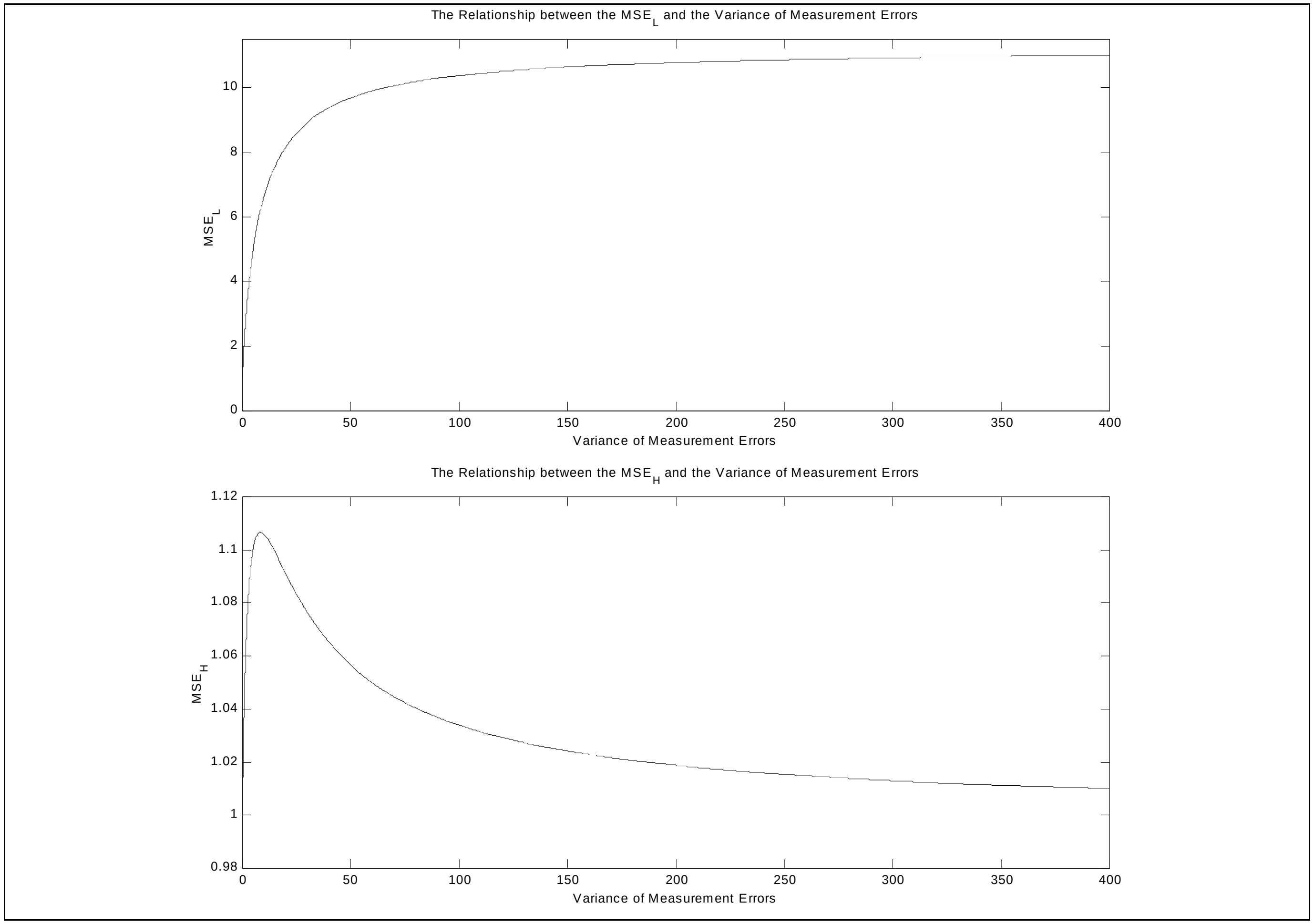


FIGURE 8. MEAN SQUARE ERRORS OF GROUP L AND GROUP H